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Topological equivalence in the theory of trapped ions

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Abstract. We consider an ion whose motion is confined to a plane perpendicular to a constant magnetic field. A harmonic trapping field is applied within the plane of the ion's motion, together with some unspecified irradiating probe beam. By determining the dynamics of the ion we show that the particle theory is formally equivalent to that of a coupled massive $(1+2)$-dimensional vector Boson field. The field characteristics of current and Proca mass take on the rôles of the interaction and harmonic potential respectively of the ion. The topological mass appropriate to the restricted configuration space, which in the case of the vector Boson field is given by the Chern–Simons term, is provided by the constant magnetic field.

1. Introduction

Theoretical studies of systems confined to $(1+2)$ dimensions, that is one temporal and two spatial coordinates, reveal many novel quantum mechanical features. At the heart of the theory lie Lagrangians that exhibit special structures: topological or Chern–Simons terms that are available only in odd-dimensional space–times. In the case of particle systems, the form of a Chern–Simons term is similar to a gyroscopic term, and involves the product of a coordinate and a velocity. It is well known that gyroscopic terms often arise in the theory of vibrations about steady motion or of vibrations involving moving constraints [1].

In the past, interest in theories relating to Chern–Simons terms was largely confined to massive gauge fields, exact quantum-gravity solutions or quantum states whose statistics are those of neither Bosons nor Fermions. General discussions and reviews are given in [2–4]. These somewhat abstract ideas were later seen to yield effective field theories for important real systems, such as strings, fibres, membranes and surfaces, where one or more spatial dimensions are either absent from the theory or do not play a significant rôle. Thus it is that the quantum Hall effect and high T_c superconductivity—two $(1+2)$-dimensional phenomena—are described in terms of Chern–Simons theory [5]. There now exists the possibility that some of the more exotic predictions of Chern–Simons theory, such as the appearance of Fermion-like properties, may have experimentally observable consequences in the behaviour of cooled atoms or ions.

In the case of an atom, the Chern–Simons term is provided by the Röntgen energy [6]. This energy is linear in the velocity of the atomic centre of mass, and its gyroscopic properties are ensured by suitable arrangements of static electric and magnetic fields. The problem is effectively turned into one analogous to a single charge moving in a constant magnetic field [7]. Certainly, a charged particle in an

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external magnetic field is the simplest real system that is describable by Chern–Simons theory, a field-theoretic relative of the Aharonov–Bohm effect [8], and, as such, provides the basis of our generic model for a particle Chern–Simons theory.

The purpose of the present paper is to draw attention to the existence of an equivalence between the dynamics of a Chern–Simons particle, such as an ion in a magnetic field, and the dynamics of a massive $(1 + 2)$-dimensional vector Boson field. Such equivalence, whereby once a property is established in one system the same property is expected in the other, may be crucial in determining integrability.

The motion of the ion in the direction of the magnetic field is that of a plane wave, and it decouples from the remaining motion in two spatial dimensions. The particle problem therefore becomes one defined in a reduced space–time of $(1 + 2)$ dimensions. The presence of a harmonic trapping potential is assumed, as, too, is some form of external radiation beam which interacts with the internal degrees of freedom of the particle. The exact natures of the beam and interaction need not be specified. The equations of motion are determined from the Lagrangian in the usual fashion and compared to those obtained in the case of a $(1 + 2)$-dimensional vector Boson field.

The trapping potential allows the theory to be satisfactorily generalized to the so-called pure Chern–Simons limit. In this limit, the Lagrangian's secular term, its kinetic energy in the particle case, is assumed to be small in relation to the other terms and is therefore ignored. When this happens the nature of the dynamics is radically altered. In particular, the eigen value of the ion's orbital angular momentum operator goes over to half-integral multiples of $\hbar$ [9].

In the particle theory, the Chern–Simons limit is obtained formally by allowing the particle mass M to vanish in the Lagrangian. This eliminates the kinetic energy or secular term from the Lagrangian. However, it is important not to confuse a vanishingly small kinetic energy with the absence of a dynamical mass, which we define as the quotient of momentum and velocity. A finite trapping potential ensures that a particle system possesses finite dynamical mass, even in the pure Chern–Simons limit, and, accordingly, a finite total energy [10]. In the field theory, the Chern–Simons limit is obtained by allowing a particular factor to vanish. This factor occurs in the Langrangian as a multiplier m of the secular term. A value of $m = 1$ produces the usual theory, whereas $m = 0$ eliminates the secular term and activates the pure Chern–Simons theory. As well as assuming the rôle of a switch, with the values of 1 (on) and 0 (off), in the manner of m, the particle mass M also performs its usual duties as a phenomenological scaling parameter—something which m is not called upon to do. It is as if two factors, one a switch and the other a phenomenological scalar, were subsumed into a single parameter M. This is perhaps an obvious point, but one which is nevertheless worth making in order to avoid the error in thinking that in the pure Chern–Simons limit the particle's dynamical mass—its 'real mass'—has somehow vanished.

At this stage, it is also worth emphasizing the differences in the several uses we make of the word *mass* in this paper. As well as the uses mentioned above, the word also occurs in the terms 'topological mass' and 'Proca' or 'conventional mass'. The first indicates the mass-like nature of the Chern–Simons term, which is apparent in the equations of motion of the field-theory potentials. On the other hand, the field theory is also made conventionally massive—something, which, unlike a topological mass, could be done in an even number of space–time coordinates. A conventionally massive field is also referred to as a Proca field [11].

2. Particle and field theories

In this section the formalisms associated with the $(1+2)$-dimensional particle and field theories are sketched, with the intention of drawing out the equivalence between their respective equations of motion. The mathematical details of the field theory are confined to an appendix. Although the convention of summing over repeated indices is used throughout this paper, an explicitly covariant notation is adopted only in the field theory; in the particle formalism no meaning is attached to the typographical position of an index.

We first consider the dynamics of a particle of mass M trapped in a harmonic potential, whose gross-motions of interest are restricted to a plane. The particle's position is therefore defined by the two-dimensional vector (R^1, R^2) and its dynamics are described by the generic Lagrangian

$$L = (M/2)(\partial_t R^\imath)(\partial_t R^\imath) + (\Theta/2)\epsilon^{\imath\jmath}R^\imath\partial_t R^\jmath - (S^2/2)R^\imath R^\imath + L_{\text{int}}, \tag{1}$$

where $\imath = 1,\ 2$. The strengths of the harmonic trapping potential and the Chern–Simons term are represented by the parameters S, Θ respectively. In the case of an ion of charge e, the magnetic field of the problem would be given by Θ/e. We note that the angular momentum of the particle arises from a finite Θ. Although $R^\imath$ is considered here to be solely a function of time, the partial-derivative notation $\partial_t = (\partial/\partial t)$ is used so as to make immediate the similarities between the particle and field theories. We have also used the two-dimensional Levi–Civita symbol, which vanishes except for the values $\epsilon^{12} = 1 = -\epsilon^{21}$. Finally, the atom's internal degrees of freedom are coupled to some light-beam probe, represented by L_{int}.

The Lagrangian (1) yields

$$(M\partial_t^2 + S^2)R^\imath - \Theta\epsilon^{\imath\jmath}\partial_t R^\jmath = \partial_\imath L_{\text{int}} \tag{2}$$

as the equation of motion, where $\partial_\imath$ denotes the differential operator $(\partial/\partial R^\imath)$. One may think of the right hand side of equation (2) as representing some form of current $J = \partial_\imath L_{\text{int}}$. If (2) is now differentiated again with respect to time we obtain

$$(M^2\partial_t^2 + \Theta^2 + MS^2)\partial_t^2 R^\imath + \Theta S^2\epsilon^{\imath\jmath}\partial_t R^\jmath = M\partial_t^2 J^\imath + \Theta\epsilon^{\imath k}\partial_t J^k, \tag{3}$$

where the identity $\epsilon^{\imath\imath'}\epsilon^{\imath'\jmath} = -\delta^{\imath\jmath}$ has been used. It is this equation for which we will now seek an equivalence in the field theory formalism.

Consider a $(1+2)$-dimensional Boson field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ minimally coupled to some current. This is the simplest field theory that can exhibit a Chern–Simons term. The oscillations of the field are made massive by the introduction of the Proca term. We will see that a conventionally massive field is necessary in order to mimic the trapping potential of the particle. The theory will be constructed from a Lagrangian, consistent with the Lorentz gauge, where the potentials A^μ are functions of $(1+2)$-dimensional space–time, characterized by the metric $(+,-,-)$. In rationalized Heaviside–Lorentz units the total Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\text{SEC}} + \mathcal{L}_{\text{PROC}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{I}} \tag{4}$$

is made up from the following four terms. The usual secular component

$$\mathcal{L}_{\text{SEC}} = -(m/4)F^{\mu\nu}F_{\mu\nu} \tag{5}$$

is switchable to an on (off) state by choosing the value of 1 (0) for the parameter m. This will allow the pure Chern–Simons limit to be selected with $m = 0$. Setting

$m = 1$ switches equation (5) to the Lagrangian density for a $(1 + 2)$-dimensional electromagnetic field. The term

$$\mathcal{L}_{\mathrm{PROC}} = (s^2/2)A^\mu A_\mu \tag{6}$$

introduces a conventional or Proca mass s into the theory. In the present scheme, the dimensions of s are the inverse of length. The Chern–Simons component

$$\mathcal{L}_{\mathrm{CS}} = (\theta/4)\epsilon^{\mu\nu\eta}F_{\mu\nu}A_\eta \tag{7}$$

is parametrized by θ, whose dimensions are the same as those of s. Finally the fields are coupled to a conserved current, where

$$\mathcal{L}_{\mathrm{I}} = -j^\mu A_\mu \tag{8}$$

is consistent with the usual minimal-coupling prescription.

The equations of motions are determined from the Lagrangian (4) in the usual manner, and from these one finds

$$\{m^2 \Box + \theta^2 + ms^2\} \Box A_\nu - \theta s^2 \epsilon_{\nu\alpha\beta}\partial^\alpha A^\beta = m \Box j_\nu - \theta\epsilon_{\nu\mu\eta}\partial^\mu j^\eta, \tag{9}$$

where $\Box = \partial_\mu\partial^\mu$ is the D'Alembertian. The details of the derivation of (9) are given in the appendix. Equation (9) describes the propagation of the fields through $(1 + 2)$-dimensional space–time. It reduces to a standard result in the special cases of a massless vector Boson field $m = 1$, $s = 0$ (the electromagnetic field) [2]. The occurrence of the parameter θ in equation (9) indicates the 'massive nature' of the Chern–Simons term. It should be compared to the conventionally massive term, the Proca mass, indicated by the parameter s. Under a gauge transformation the topological mass—the Chern–Simons term equation (7)—transforms as a total derivative. On the other hand, the conventional mass—the Proca term equation (6)—is gauge non-invariant.

Equation (9) is the field equivalent to the particle propagation equation (3). The difference in sign between two pairs of terms is because equation (9) is written in a manifestly covariant notation while equation (3) is not. The terms linear in θ in equation (9) are 'twisted' transformations of potentials and currents. These are mirrored in equation (3) by similar transformations of the spatial coordinates R^i and 'currents' J. The occurrence of these types of transformation is a characteristic of Chern–Simons theory—this is particularly evident in the pure Chern–Simons limits ($M = 0$, $m = 0$) of equations (9) and (3). They give, in general, a trochoidal shape to the ion's trajectory, as is well known in the $S = 0$ case [12].

3. Conclusions and discussion

We have demonstrated that the dynamics of an ion moving in a magnetic field and harmonic trapping potential are formally equivalent to a $(1 + 2)$-dimensional massive vector Boson field. The parallel features of the particle and field theories are tracked in equations (3) and (9). The constant magnetic field assumes the rôle of the topological mass of the vector Boson case, with a direct correspondence between the parameters θ and Θ. Similar equivalencies exist between the Proca mass term s and the strength of the trapping potential S.

If the atom interacts with an external probe beam then the equivalence is maintained by coupling the Boson field to a current. The gradient of the interaction in the particle theory is equivalent to the conserved current of the field theory.

The formal equivalence between the field and particle theories is consistent with the suggestion that Chern–Simons effects might translate into observable consequences for the behaviour of cooled atoms and ions. Of these effects, the appearance, in the pure Chern–Simons limit, of a half-integral spectrum for the canonical orbital angular momentum operator [9] is perhaps the most novel. Although this spectrum cannot itself be observed, its half-integral nature may nevertheless be a characteristic of an as yet unknown mechanism, one which conceivably possesses observable features.

To give some idea of the feasibility of ignoring the kinetic energy term from the Lagrangian we choose a Mg^+ ion with an orbital radius of about a micron in a magnetic field of about 5 T. In this case, the ion must be cooled to speeds that are significantly less than $20\,\mathrm{m\,s^{-1}}$ if the Lagrangian's kinetic energy—the secular term—is to be ignored. The attainment of such temperatures is consistent with current experimental development; however, whether or not the pure Chern–Simons limit is accessible in the simple and specific case of an ion in a magnetic field is an open question.

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Appendix

In this appendix the mathematical details associated with the derivation of equation (9) are given. Use is made of the identity

$$\epsilon^{\mu\nu\eta}\epsilon_{\mu'\nu'\eta} = \delta^{\mu}_{\mu'}\delta^{\nu}_{\nu'} - \delta^{\mu}_{\nu'}\delta^{\nu}_{\mu'} \tag{A 1}$$

involving the three-dimensional Levi–Civita tensor. This tensor vanishes except for cyclic variations of $\epsilon^{012} = -\epsilon^{102} = 1$.

The equations of motion

$$m\partial_\mu F^{\mu\eta} + s^2 A^\eta + (\theta/2)\epsilon^{\eta\mu\nu}F_{\mu\nu} = j^\eta \tag{A 2}$$

are determined from the total Lagrangian (4) by use of the Euler–Lagrange equations. To see how the potentials behave dynamically it is useful to introduce the dual field

$${}^\star F^\eta = (1/2)\epsilon^{\eta\mu\nu}F_{\mu\nu}, \tag{A 3}$$

which obeys the Bianchi relationship

$$\partial_\mu {}^\star F^\mu = 0. \tag{A 4}$$

From the identity (A 1) it is evident that the inverse of (A 3) is

$$F^{\mu\nu} = \epsilon^{\mu\nu\eta}\,{}^\star F_\eta. \tag{A 5}$$

Note the absence of the factor 1/2 in equation (A 5). A direct substitution of (A 3) into (A 2) gives

$$m\partial_\mu F^{\mu\eta} + s^2 A^\eta + \theta\, {}^\star F^\eta = j^\eta. \tag{A 6}$$

Equation (A 6) may be recast into the form

$$m(\partial_\mu\, {}^\star F_\nu - \partial_\nu\, {}^\star F_\mu) - \theta F_{\mu\nu} = -\epsilon_{\mu\nu\eta} j^\eta + s^2 \epsilon_{\mu\nu\eta} A^\eta \tag{A 7}$$

by means of (A 1), (A 3) and (A 5). To achieve (A 7), re-form the first term of (A 6) into the dual, by means of (A 5), and then multiply throughout by a four-dimensional Levi–Civita tensor. This gives

$$m\partial_\mu \epsilon_{\beta\lambda\eta} \epsilon^{\mu\alpha\eta}\, {}^\star F_\alpha - s^2 \epsilon_{\beta\lambda\eta} A^\eta - \theta \epsilon_{\beta\lambda\eta}\, {}^\star F^\eta = -\theta \epsilon_{\beta\lambda\eta} j^\eta, \tag{A 8}$$

after changing the order of the superscript indices on one of the Levi–Civita tensors from $\mu\eta\alpha$ to $\mu\alpha\eta$ and then multiplying throughout by -1. If now the last term on the left-hand side of (A 8) is re-written from ${}^\star F^\eta = (1/2)\epsilon^{\theta\phi\eta} F_{\theta\phi}$ and a summation over η is performed then (A 7) is obtained after the use of the identity (A 1). By taking the divergence of (A 7), and using the Bianchi relationship (A 4), one obtains

$$m \,\Box\, {}^\star F_\nu = \theta \partial^\mu F_{\mu\nu} + \epsilon_{\nu\mu\eta} \partial^\mu j^\eta - s^2 \epsilon_{\nu\mu\eta} \partial^\mu A^\eta \tag{A 9}$$

after changing the order of the indices on the three-dimensional Levi–Civita symbol. A substitution from (A 6) for the dual-field term

$$\theta\, {}^\star F_\nu = j_\nu - s^2 A_\nu - m \,\Box\, A_\nu + m \partial_\nu \partial_\alpha A^\alpha \tag{A 10}$$

enables one to re-write equation (A 9) as equation (9) of section 2.

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