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Abstract

An ion moving in a constant magnetic field is considered as the generic example of two-dimensional topological (Chern-Simons) theory. The eigenfunctions for the generic particle are determined by solving Schrödinger's equation in the secular and non-secular forms of the theory, corresponding to the inclusion and non-inclusion respectively in the Lagrangian of the particle's kinetic energy. A notable feature is the self-consistency of the formalism in the limit of the non-secular theory: the eigenfunction remains well-behaved and single-valued, and the eigenenergy does not diverge. The non-secular eigenfunction is consistent with a half-integral angular momentum spectrum. It is shown that the non-secular regime is unlikely to be achieved by a naïve cooling of the ion. A suggestion is made that the non-secular regime might be experimentally accessible through Laguerre-Gaussian optics.

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I. INTRODUCTION

Theoretical studies of systems confined to $(1 + 2)$ dimensions, that is one temporal and two spatial co-ordinates, reveal many novel quantum mechanical features. At the heart of the theory lie Lagrangians that exhibit special structures: topological terms that are available only in odd-dimensional spacetimes. It has been speculated that some of the more exotic predictions of the theory, such as the appearances of Fermion-like properties, may have experimentally observable consequences in the behaviour of cooled atoms or ions [1]. The motivation behind the present paper is to investigate this proposition further by determining the eigenfunction and eigenenergy spectrum of a suitable particle in the presence of an external trapping potential. This is undertaken in two regimes of the theory, corresponding respectively to the inclusion and exclusion of the kinetic energy from the Lagrangian. The experimental arrangement envisaged is in principle that of an trapped ion or atom, cooled to a sufficiently low temperature that its kinetic energy may be assumed to be negligibly small compared to the other terms in the Lagrangian. The theory is developed for a single-particle system — which is regarded as the generic example — where the form of the topological term in the Lagrangian is similar to a gyroscopic term, involving the product of a co-ordinate and a velocity.

It has long been known that gyroscopic terms often arise in Lagrangians that describe vibrations about some steady motion or vibrations involving moving constraints [2]. The basis of these simple mechanical ideas was generalised by Chern and Simons, who in 1971 showed that certain fibre bundles in Riemann space remained unchanged under a conformal transformation of the space's metric [3]. Gyroscopic terms involve cross-products of position and velocity: they are particle-like specialisations of topological Lagrangians, which, under the sobriquet of Chern-Simons, are familiar in odd-dimensional spacetime field-theory. At first, interest in theories relating to Chern-Simons terms was largely confined to massive gauge fields, exact quantum-gravity solutions or quantum states whose statistics are neither those of Bosons nor Fermions. General discussions and reviews are given in [4], [5]. Such notions were later seen to yield effective field theories for important real systems of strings, fibres, membranes and surfaces, where one or more spatial dimensions are either absent from the theory or do not play a significant rôle. Thus it is that the quantum Hall effect and high T_c superconductivity — two $(1+2)$-dimensional phenomena — may be describable in terms of Chern-Simons theory [6]. Chern-Simons theory has also been regarded as a field-theoretic relative of the Aharonov-Bohm effect [7].

The quantum behaviour of atoms and ions confined to two dimensions exhibit characteristics arising through the presence of gyroscopic terms in the Lagrangian. In the case of a trapped ion in a constant magnetic field, the gyroscopic term simply arises as a consequence of the Lorentz interaction. In the case of an atom, the gyroscopic term is provided by the Röntgen energy [1], [8]. This energy is linear in the velocity of the atomic centre-of-mass, and its gyroscopic properties are ensured by suitable arrangements of static electric and magnetic fields. The problem is effectively turned into one analogous to a single charge moving in a constant magnetic field [9].

In the present paper, a charged particle in a constant external magnetic field provides the basis of the generic-particle model of the theory. The motion of the ion in the direction of the magnetic field is that of a plane wave, which decouples from the remaining motion in two spatial dimensions. The generic problem therefore becomes one defined in a reduced spacetime of $(1+2)$ dimensions. The presence of an external trapping potential supplies the effects of a dynamical mass — the quotient of momentum and velocity — when the kinetic energy is neglected. This is achieved formally by allowing the particle mass M to vanish from the theory, eliminating the kinetic energy — the so-called secular term — from the Lagrangian. The following semantics are introduced. When the generic particle Lagrangian contains the kinetic energy — the secular term — the theory is said to describe the *secular* regime. Likewise, the *non-secular* regime is said to be operative when the kinetic energy is absent from the Lagrangian. The transition from the secular to non-secular regimes is termed the *non-secular limit.* By way of a reminder, these definitions will be given again at appropriate junctures. Secular and non-secular are chosen in preference to terms involving the word *kinetic* so as not to prejudice field theory in, for example, the case of a $(1+2)$-dimensional vector Boson field [10].

There are some similarities between the secular and non-secular regimes and the so-called *full* and *reduced* regimes of Chern-Simons theory [11]. The reduced theory is sometimes referred to in the literature as *pure* Chern-Simons theory. However, the reader is cautioned to be aware that these similarities are potentially confusing since there are important differences between the forms of the usual particle analogue of Chern-Simons theory and the system presented in the present paper. In the former, an harmonic oscillator is considered in the presence of a gyroscopic (Chern-Simons) term. The potential energy of the oscillator, which is directly proportional to the particle mass, is introduced in order to mimic the Higgs mechanism of incorporating the effects of a mass into a gauge field. The pure Chern-Simons limit — like the non-secular limit — corresponds to putting the particle mass to zero, but its effect is that *both* the kinetic *and* potential energies of the oscillator vanish. This is obviously not the same as the generic particle model of the present paper, where the harmonic trapping-potential energy is considered to be *externally* supplied by some configuration of electromagnetic fields, and, therefore, does not vanish in the non-secular limit. In the pure Chern-Simons limit, the eigenfunction becomes intrinsically one-dimensional. This cannot occur in the generic particle model, where a physical particle sits in a real non-vanishing external potential. The eigenfunction remains two-dimensional in the non-secular limit; the component in the direction of the magnetic field — the third dimension — is that of a plane wave and is ignored.

The gyroscopic and harmonic trapping-potential energies are assumed to extend throughout the two-dimensional space; any boundaries to the system are therefore considered to be infinitely removed. The effects of this are that there is no limit to the angular momentum quantum number and Aharonov-Bohm phase shifts are absent. The latter would occur if the gyroscopic term were to be effectively reduced to that of a positional delta-function. The eigenfunction of the generic particle is determined in the present paper by solving Schödinger's equation in polar co-ordinates, but generalised in such a way that allows the non-secular limit to be reached in a consistent and well-behaved manner. The formalism is straightforward: it is nevertheless non-standard. Thus, the calculations are described in some detail.

The results of the present paper suggest that the non-secular limit is unlikely to be experimentally observed by simply cooling the generic particle to a point where its kinetic energy is negligibly small. This implies that it is not possible to achieve the non-secular limit by physical means, which is consistent with the notion that the limit is not one in any smooth transitional sense. However, any suggestion of its inaccessibility to experiment raises questions of how the limit is to be interpreted, since the calculations of the eigenfunctions and energy spectra are compliant with the tenets of non-relativistic quantum mechanics. Such questions remain to be answered.

The contents of the paper are laid out as follows. The theory is introduced in the next section, where the terminology is defined. The secular and non-secular regimes are considered in Section III and their angular momentum properties in Section IV. In Section V, the method used for the determination of the general eigenfunction is described. Also in this section, it is shown that the eigenenergy does not diverge. The theory is then specialised to the secular (Section VI) and non-secular (Section VII) regimes, with the mathematical details gathered into two appendices. The connection between polar and Cartesian eigenfunctions, reflecting the relationship between the non-secular theory and the standard theory of the one-dimensional harmonic oscillator is discussed in Section VIII. Finally, the conclusions are delineated and discussed in Section IX.

II. GENERIC PARTICLE THEORY

In this section, the generic particle system is defined in terms of its Lagrangian. The canonical procedure is used to determine the system's quantum mechanical Hamiltonian. The energy spectrum is equivalent to that of an anisotropic oscillator.

Consider the dynamics of a particle of mass M trapped in an harmonic potential, whose gross-motions of interest are restricted to a plane. The particle's position is therefore defined by the two-dimensional vector $\mathbf{R} = (R_1, R_2)$ and the dynamics are described by the generic Lagrangian

$$L = (M/2)\dot{R}_\imath \dot{R}_\imath + (\Theta/2)\epsilon_{\imath\jmath} R_\imath \dot{R}_\jmath - (S^2/2) R_\imath R_\imath, \tag{1}$$

where $\imath = 1,\ 2$. In the present paper, a summation over repeated indicies is implicit in the formalism and an overdot signifies temporal differentiation. Use is made of the two-dimensional Levi-Civita symbol; a quantity which vanishes for all values except $\epsilon_{12} = 1 = -\epsilon_{21}$. The strengths of the harmonic trapping potential and the Chern-Simons term are represented by the parameters S, Θ respectively. In the case of an ion of charge e, the magnetic field of the problem would be given by Θ/e.

It might seem that the parameter Θ could assume negative as well as positive values, which, in the example of an ion in a magnetic field, would correspond to a reversal in the direction of the magnetic field. However, a change in the sign of Θ only changes the sign of the angular momentum eigenvalue in the non-secular limit; it does not materially alter the physics of the problem. Consequently, it will be assumed that Θ is a positive quantity in all that follows.

The classical equations of motion stemming from the generic Lagrangian (1) are determined by Lagrange's equation to be

$$M\ddot{R}_\imath + S^2 R_\imath - \Theta \epsilon_{\imath\jmath} \dot{R}_\jmath = 0. \tag{2}$$

They are discussed elsewhere [10]: the main purpose of the present paper is a delineation of the consequences of the quantised theory. A canonical quantisation of the dynamics of the generic system leads from the conjugate linear momentum

$$P_\imath = M\dot{R}_\imath - (\Theta/2)\epsilon_{\imath\jmath} R_\jmath \tag{3}$$

to the Hamiltonian operator

$$\hat{H} = \frac{\hat{P}_\imath \hat{P}_\imath}{2M} - \frac{\Theta}{2M}\epsilon_{\imath\jmath} \hat{R}_\imath \hat{P}_\jmath + \frac{M\Omega^2}{2} \hat{R}_\imath \hat{R}_\imath, \tag{4}$$

associated with the equal-time commutator

$$\left[\hat{R}_\imath, \hat{P}_\jmath\right] = i\hbar\delta_{\imath\jmath}, \tag{5}$$

where

$$\Omega^2 = \left\{\frac{\Theta}{2M}\right\}^2 + \frac{S^2}{M} \tag{6}$$

contains the phenomenological parameters of the problem. At this stage, the sign of Ω is arbitrary. Quantum mechanical operators are indicated typographically by a caret.

The system described by Eq. (1) is equivalent to two uncoupled oscillators. The Hamiltonian (4) may therefore be cast in a form

$$\hat{H} = \hbar \sum_{\alpha=-,+} \omega_\alpha \left\{\hat{a}^\dagger_\alpha \hat{a}_\alpha + (1/2)\right\} \tag{7}$$

involving the two sets ($\pm$) of Boson annihilation and creation operators

$$\hat{a}_\pm = \left[\frac{\omega_\pm}{2\hbar}\right]^{\frac{1}{2}} \hat{q}_\pm + i\left[\frac{1}{2\hbar\omega_\pm}\right]^{\frac{1}{2}} \hat{p}_\pm \tag{8a}$$

$$= \frac{1}{\sqrt{2}}\left\{\left[\frac{M\Omega}{2\hbar}\right]^{\frac{1}{2}} \left[\hat{R}_1 \pm i\hat{R}_2\right] + [2M\Omega\hbar]^{-\frac{1}{2}} \left[i\hat{P}_1 \mp \hat{P}_2\right]\right\}, \tag{8b}$$

$$\hat{a}^\dagger_\pm = \left[\frac{\omega_\pm}{2\hbar}\right]^{\frac{1}{2}} \hat{q}_\pm - i\left[\frac{1}{2\hbar\omega_\pm}\right]^{\frac{1}{2}} \hat{p}_\pm \tag{8c}$$

$$= \frac{1}{\sqrt{2}}\left\{\left[\frac{M\Omega}{2\hbar}\right]^{\frac{1}{2}} \left[\hat{R}_1 \mp i\hat{R}_2\right] - [2M\Omega\hbar]^{-\frac{1}{2}} \left[i\hat{P}_1 \pm \hat{P}_2\right]\right\}, \tag{8d}$$

defined in terms of the Cartesian components of the position $\hat{\mathbf{R}}$ and momentum $\hat{\mathbf{P}}$ operators. It will be seen in Section IV that the individual oscillator-frequencies

$$\omega_\pm = \Omega \pm \frac{\Theta}{2M} \tag{9}$$

(9) are directly related to the generic particle's free-motion angular velocities. Equation (6) indicates that the modulus of Ω must be greater than $\Theta/2M$, for a non-vanishing value for S. Therefore, adopting the oscillator frequencies $\omega_\pm$ as necessarily positive quantities, Eq. (9) implies that Ω must also be regarded as positive. Throughout the present paper,

any equation involving $\pm$ and/or $\mp$ signs — such as Eq. (9) — is to be read as a concise statement of two independent equations; one of these is formed from taking only the upper signs and the other from taking only the lower signs.

The operators (8) act on Fock states, parameterised by positive integers $n_\pm$ — the eigenvalues of the number operators $\hat{a}_\pm^\dagger \hat{a}_\pm$ — according to the usual ladder-rules

$$\hat{a}_\pm |n_\pm\rangle = \sqrt{n_\pm}|n_\pm - 1\rangle, \tag{10a}$$

$$\hat{a}_\pm^\dagger |n_\pm\rangle = \sqrt{n_\pm + 1}|n_\pm + 1\rangle, \tag{10b}$$

$$\langle n_\alpha | n_{\alpha'}\rangle = \delta_{\alpha\alpha'}, \qquad \alpha, \alpha' = \pm. \tag{10c}$$

The equal-time commutator

$$\left[\hat{a}_\alpha, \hat{a}_{\alpha'}^\dagger\right] = \delta_{\alpha\alpha'}, \qquad \alpha, \alpha' = \pm \tag{11}$$

is consistent with (5).

Equations (7), (10) lead to a straightforward determination of the eigenenergy

$$E = \hbar\Omega\left\{n_+ + n_- + 1 + \frac{\Theta}{2M\Omega}\left(n_+ - n_-\right)\right\} \tag{12}$$

of the Hamiltonian, associated with the Fock eigenstates $|n_\pm\rangle$, according to the Schrödinger's equation $\hat{H}\Psi = E\Psi$. The spectrum is composed of the so-called Fock-Darwin energy levels [12]. The parameter S vanishes in the absence of an harmonic trapping potential, with the result that

$$\Omega = \frac{\Theta}{2M}, \qquad \text{for } S = 0. \tag{13}$$

Consequently, Eq. (12) collapses to the usual Landau levels $(\hbar\Theta/M)(n_+ + [1/2])$; infinitely degenerate since, as mentioned in Section I, the parameter Θ — "the magnetic field" — is non-zero over all the two-dimensional space. For finite S, the Fock-Darwin energy levels do not converge to the Landau levels completely as the dimensionless number $\Theta/2S\sqrt{M}$ expands with increasing Θ and Ω tends to (13); instead, they form sheets of an infinite number of levels on top of each Landau level [12]. There are some apparent similarities between Landau theory and the non-secular theory to be described in the present paper. For example, Eq. (13) is suggestive of the later Eq. (24), and both theories reduce to one corresponding to a single harmonic oscillator. However, these similarities are superficial and do not appear to have wider significance. The onset of reduction to the single oscillator of Landau theory (vanishing S but finite M) will be seen to be different from that in non-secular theory (vanishing M but finite S). Again, the manner of reduction is different. Landau theory involves only the (+) oscillator and its spectrum is written in terms of the n_+ quantum number. On the other hand, non-secular theory is a reduction to a single new oscillator, one which neither corresponds wholly to the (−) nor the (+) oscillators of the secular theory. To complete this catalogue of related theories, the particle analogue of pure Chern-Simons theory may be characterised *formally* by an M and an effective S that both vanish.

III. SECULAR AND NON-SECULAR REGIMES

The presence of the kinetic energy, the secular term, in the Lagrangian (1) ensures on quantisation the canonical form (5) of the commutator between the dynamical variables. The reader is reminded that the adjective *secular* is used to denote the theory of the generic system — defined by the Lagrangian (1). This allows a distinction to be made from the non-secular theory, where the secular term is omitted and the generic Lagrangian (1) reduces to

$$L = (\Theta/2)\epsilon_{ij} R_i \dot{R}_j - (S^2/2) R_i R_i. \tag{14}$$

The transformation from secular to non-secular regimes is evoked by means of the non-secular limit $M \to 0$. Thus, for example, the classical equations of motion of the non-secular theory are those of the secular theory in the limit $M \to 0$. That is

$$S^2 R_i - \Theta \epsilon_{ij} \dot{R}_j = 0, \tag{15}$$

after putting M equal to zero in Eq. (2). Of course, Eq. (15) could equally have been obtained directly from (14) by means of Lagrange's equation.

The non-secular Lagrangian (14) is first-order: it describes a system characterised by primary constraints. The canonical quantisation prescription for constrained systems is given by Dirac [13], [14]. This is the method adopted here, although it is noted that there exists an alternative procedure that avoids the use of weak equalities [15], [16]. The primary constraints

$$\pi_\imath = P_\imath + (\Theta/2)\epsilon_{\imath\jmath}R_\jmath \approx 0, \tag{16}$$

associated with the Lagrangian (14) are indicative of the fact that the canonical momenta $P_\imath$ are not independent variables. Following the usual procedure, the "weakly equals" symbol $\approx$ indicates that the Poisson brackets of interest must be evaluated before any use is made of the constraints [13]. In other words, although $\pi_\imath$ vanishes formally it may have nonvanishing canonical Poisson brackets [17]. There are no secondary constraints since the Poisson bracket

$$\{\pi_\imath, \pi_\jmath\} = \epsilon_{\imath\jmath}\Theta \tag{17}$$

between the primary constraints is non-zero. There is no velocity term associated with the canonical momentum. This has important consequences when the non-secular theory is quantised.

One should not confuse a vanishingly-small secular term, the kinetic energy, brought about by taking the non-secular limit $M \to 0$, with the absence of what will be termed here as the *dynamical mass*, defined as

$$\mu = |\mathbf{P}|/|\dot{\mathbf{R}}| \tag{18}$$

— the ratio of the conjugate-momentum and speed. Indeed, in the classical non-secular theory the equations of motion (15) and the constraint (16) imply a non-zero value of

$$\mu = \Theta^2/2S^2 \tag{19}$$

for the dynamical mass, together with a classical force equal to $-(S^2/2)\mathbf{R}$ and a total energy — the classical Hamiltonian — of $(S^2/2)\mathbf{R}^2$. All this is consistent with the purely mechanical aspects of the quantised non-secular theory. It will be seen in Section VII that in the non-secular theory the dynamical mass μ plays a similar rôle to that of M in the secular theory.

Within the canonical quantisation prescription, the transformation of dynamical variables to quantum mechanical operators occurs after the evaluation of the Poisson brackets. The non-secular linear momentum operator may therefore be directly identified to be

$$\hat{P}_\imath = -\frac{\Theta}{2}\epsilon_{\imath\jmath}\hat{R}_\jmath. \tag{20}$$

As such, it is related to the co-ordinate position-operator in a manner quite different from that seen in the corresponding equation of the secular theory — Eq. (3). The presence of primary constraints has another important effect in the determination of the commutator

$$\left[\hat{A}, \hat{B}\right] = i\hbar\left\{A, B\right\} + \frac{i\hbar}{\Theta}\left\{A, \pi_\imath\right\}\epsilon_{\imath\jmath}\left\{\pi_\jmath, B\right\} \tag{21}$$

between any two quantum mechanical operators $\hat{A}$, $\hat{B}$ of interest. The usually-seen derivation of a commutator from the Poisson brackets is changed by the addition of the final term in Eq. (21). This results in the non-vanishing commutators

$$\left[\hat{R}_\imath, \hat{P}_\jmath\right] = i(\hbar/2)\delta_{\imath\jmath}, \tag{22a}$$

$$\left[\hat{R}_\imath, \hat{R}_\jmath\right] = -i(\hbar/\Theta)\epsilon_{\imath\jmath}, \tag{22b}$$

$$\left[\hat{P}_\imath, \hat{P}_\jmath\right] = -i\hbar(\Theta/4)\epsilon_{\imath\jmath} \tag{22c}$$

between the conjugate variables. Note particularly the presence of the factor of one-half in Eq. (22a), and the non-vanishing forms of Eqs. (22b), (22c). These unusual values for the commutators between the quantum mechanical operator-variables in the non-secular regime have no equivalents in the classical theory. There is no difference between the Poisson brackets of the conjugate variables $\mathbf{R}$, $\mathbf{P}$ in non-secular regime compared to those of the secular regime.

The two sets of conjugate operators $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ are not independent in the non-secular theory and the Hamiltonian operator

$$\hat{H} = \frac{S^2}{2}\hat{\mathbf{R}}^2 = \frac{2S^2}{\Theta^2}\hat{\mathbf{P}}^2 \tag{23}$$

may be expressed in either variable. The Hamiltonian represents a single oscillator: each variable ($\hat{\mathbf{R}}$ or $\hat{\mathbf{P}}$) may be written directly as a linear combination of Bosonic annihilation and creation operators, consistent with (22), such that (23) reduces to an immediately apparent oscillatory form — see Eq. (25), below. It is instructive, however, to achieve this from the prospect of the secular regime, by evoking the non-secular limit. From Eq. (6),

$$M\Omega \to \frac{\Theta}{2} \text{ as } M \to 0. \tag{24}$$

That is: although M vanishes in the non-secular limit, the product $M\Omega$ remains finite. The effect of (24) is to reduce the number of oscillators of the theory from two in the secular regime to one in the non-secular regime. The (+) oscillator is not present in the theory of the non-secular regime, in the sense that when Eq. (24) is substituted into (8), and Eq. (20) is used, the operators $\hat{a}_+$, $\hat{a}_+^\dagger$ vanish. Using the notation $\lim_{M\to 0}$ to indicate the effects of Eq. (24), the non-secular Hamiltonian

$$\hat{H} = \hbar\bar{\omega}\left\{\hat{a}^\dagger\hat{a} + \frac{1}{2}\right\} = \frac{\Theta\bar{\omega}}{2}\hat{\mathbf{R}}^2, \tag{25}$$

where

$$\bar{\omega} = S^2/\Theta, \tag{26}$$

may be written in terms of the non-secular limits

$$\begin{aligned}\hat{a} = \lim_{M\to 0}\hat{a}_- &= \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\left(\hat{R}_1 + \frac{2i}{\Theta}\hat{P}_1\right) = \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\left(\hat{R}_1 - i\hat{R}_2\right)\\ &= \left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\left(\hat{P}_2 + \frac{i\Theta}{2}\hat{R}_2\right) = \left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\left(\hat{P}_2 - i\hat{P}_1\right),\end{aligned} \tag{27a}$$

$$\begin{aligned}\hat{a}^\dagger = \lim_{M\to 0}\hat{a}_-^\dagger &= \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\left(\hat{R}_1 - \frac{2i}{\Theta}\hat{P}_1\right) = \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\left(\hat{R}_1 + i\hat{R}_2\right)\\ &= \left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\left(\hat{P}_2 - \frac{i\Theta}{2}\hat{R}_2\right) = \left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\left(\hat{P}_2 + i\hat{P}_1\right)\end{aligned} \tag{27b}$$

of the $(-)$ operators (8). Equation (20) has been used to express Eq. (27) in all equivalent forms involving the Cartesian components of $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$. The operators (27) are consistent with the commutators (22) and

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = 1. \tag{28}$$

They are therefore Bosonic. Equation (25) arises from that portion of the two-oscillator Hamiltonian (7) remaining in the non-secular limit, where, from (6) and (9), $\lim_{M\to 0}\omega_{-} = S^2/\Theta$. It is the same Hamiltonian as (23): the non-secular Hamiltonian derived directly from the non-secular Lagrangian (14), according to the canonical quantisation prescription for constrained systems. Of course, the operators $\hat{a}$, $\hat{a}^{\dagger}$ could have been introduced directly into (23) by re-arranging (27) into explicit equations in $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$.

The word *limit* might suggest a smooth transitional process. This is not the case in the non-secular limit, and it is important to appreciate that the operators $\hat{a}$, $\hat{a}^{\dagger}$ represent a different oscillator from that represented by $\hat{a}_{-}$, $\hat{a}_{-}^{\dagger}$. This difference is not just phenomenological, such as a simple re-scaling of frequencies; it is one in which the nature of the quantum dynamical variables $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ undergoes a radical shift. The nature of $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ in the secular regime is revealed by the independence of the operators, one from the other, and the value of the fundamental commutator (5). However in the non-secular regime, the operators are no longer independent variables — their inter-dependence is manifested through Eq. (20) — and the value of their fundamental commutator is now changed to that given by Eq. (22a). The radical shift in the nature of $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$ in the non-secular limit is carried by the non-secular operators $\hat{a}$, $\hat{a}^{\dagger}$ even though the value of their commutator remains appropriate for Boson operators. It may therefore be misleading to suppose that in the non-secular limit one of the oscillators vanishes, for this implies that the other oscillator remains *in situ*. A more accurate picture would be that both oscillators $\hat{a}_{\pm}$, $\hat{a}_{\pm}^{\dagger}$ disappear in the non-secular limit; to be replaced by a single new oscillator $\hat{a}$, $\hat{a}^{\dagger}$.

The non-secular regime eigenenergy spectrum

$$E = \hbar\bar{\omega}\left\{m + \frac{1}{2}\right\} \tag{29}$$

follows from the Hamiltonian (25). Equation (29) is written in terms of the integer eigenvalue m of the number operator $\hat{a}^\dagger \hat{a}$ — the parameter of the Fock states $|m\rangle$ that are generated according to the usual ladder-rules by $\hat{a}$, $\hat{a}^\dagger$, after the manner of (10). This is a new quantum number: it is not the eigenvalue of the number operator $\hat{a}_-^\dagger \hat{a}_-$; indeed, the divergent non-secular limit

$$\lim_{M\to 0} \hbar\Omega \left\{ n_+ + n_- + 1 + \frac{\Theta}{2M\Omega}(n_+ - n_-) \right\} = \lim_{M\to 0} \hbar \left\{ \frac{\Theta}{M} + 2\bar{\omega} \right\} \left[n_+ + \frac{1}{2} \right] \tag{30}$$

of the eigenenergy (12) is not even a function of n_-.

How can the divergent non-secular limit (30) be reconciled with the well-behaved spectrum (29)? The heart of the problem lies in applying the non-secular limit to an eigenenergy in isolation, detached from its Hamiltonian. Care needs to be taken when considering an eigenvalue in isolation; particularly so when determining limiting values. An eigenvalue is only one component of the formalism: the other component is the corresponding eigenfunction; the bridge between the two is of course the eigenvalue equation. The Hamiltonian and eigenenergy must remain together in a properly constructed Schrödinger equation, and the present paper shows that if the non-secular limit is made at the level of the Schrödinger equation then an eigenenergy spectrum is produced that does not diverge in the non-secular limit. This is presented in Section V.

In discussing the divergent nature of the eigenenergy (30) one should be aware of an early investigation [18] involving a particle anologue of Chern-Simons theory, where the potential energy appears to have the same form as in Eq. (1). A seemingly *ad hoc* attempt is made to remove the divergent term from an equation similar to (30) by an infinite subtraction. The exact mechanisms involved is not specified. Further, because of the factor of 2 multiplying $\bar{\omega}$ in the right-hand side of (30), a further subtraction — this time, finite — must be made in order to bring it into line with (29). The authors admit that this has no *a priori* theoretical basis. The problem does not arise in later particle analogues of Chern-Simons theory, where the potential energy vanishes in the pure Chern-Simons limit [11].

IV. ANGULAR MOMENTUM

The main feature of angular momentum in Chern-Simons theory is the abrupt change in the nature of the eigenvalue spectrum in the pure Chern-Simons limit. A similar feature occurs in the present theory. An integer spectrum prevails in the secular regime, characteristic of the Boson nature of the quantum-particle oscillations. In the non-secular regime the spectrum is one of half-integers and the particle behaves, in the context of angular momentum, as if it were a Fermion. These angular momentum characteristics will now be reviewed: they have been discussed more fully elsewhere [1].

Angular momentum is the generator of rotation. It is therefore convenient and natural to introduce polar co-ordinates

$$\rho = \left| \left\{ R_1^2 + R_2^2 \right\}^{\frac{1}{2}} \right|, \tag{31a}$$

$$\phi = \tan^{-1} \left\{ \frac{R_1}{R_2} \right\} \tag{31b}$$

in which to express the classical angular momentum

$$p_\phi = \epsilon_{\imath\jmath} R_\imath P_\jmath = M\rho^2\dot{\phi} + (\Theta/2)\rho^2 \tag{32}$$

of the generic particle. It is apparent from (32) that the classical angular momentum reduces to $(\Theta/2)\rho^2$ in the non-secular limit. The topological feature of the system, characterised by Θ, modifies the usually-seen identification $M\rho^2\dot{\phi}$ of the conjugate angular momentum p_ϕ by the addition of the term $(\Theta/2)\rho^2$. This term modifies the areal velocity

$$\rho^2\dot{\phi} = -(\Theta/2M)\rho^2 + \text{a constant} \tag{33}$$

from the more usual value associated with central forces. Equation (33) is obtained by an application of Lagrange's equation to the polar form

$$L = (M/2)\dot{\rho}^2 + U_{\text{effective}} \tag{34}$$

of the Lagrangian (1), where $U_{\text{effective}}$ is the effective potential energy. This takes on the form

$$U_{\text{effective}} = \frac{M\rho^2}{2}(\dot{\phi} - \omega_-)(\dot{\phi} + \omega_+) \tag{35}$$

when written in terms of Eq. (9). Thus, the particle's motion is free in the secular regime when $\dot{\phi}$ is either equal to ω_- or $-\omega_+$. The non-secular regime Lagrangian

$$L = \frac{\Theta\rho^2}{2}(\dot{\phi} - \bar{\omega}), \tag{36}$$

where Eq. (26) has been used, vanishes when $\dot{\phi} = \bar{\omega}$. Of course, the corresponding Hamiltonian does not vanish: it always has a non-secular value equal to the product of $\bar{\omega}$ and the angular momentum, as indicated below in Eq. (40).

The angular momentum operator

$$\hat{p}_\phi = \epsilon_{\imath\imath'} \hat{R}_\imath \hat{P}_{\imath'} \tag{37}$$

is a constant of the motion in both the secular and non-secular regimes. However, its eigenvalue spectra are radically different. Using Equation (8), the angular momentum operator

$$\begin{aligned} \hat{p}_\phi &= (\hbar/2)\left\{\hat{a}_-^\dagger \hat{a}_- + \hat{a}_- \hat{a}_-^\dagger\right\} - (\hbar/2)\left\{\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_+ \hat{a}_+^\dagger\right\} \\ &= \hbar\left\{\hat{a}_-^\dagger \hat{a}_- + (1/2)\right\} - \hbar\left\{\hat{a}_+^\dagger \hat{a}_+ + (1/2)\right\} \end{aligned} \tag{38}$$

of the secular theory can be re-cast in terms of the Boson operators introduced in Section II. The angular momentum eigenvalue spectrum

$$p_\phi = \hbar(n_- - n_+) \tag{39}$$

in the secular regime follows from (38). Its non-zero portion consists of positive and negative integers; a state of affairs which is brought about formally by the cancellation of factors of one-half in the two terms on the right of Eq. (38). The integers $n_\pm$ are associated with the Fock states of the operators $\hat{a}_\pm$, $\hat{a}_\pm^\dagger$, in the manner defined in Eq. (10). This is to be contrasted with the determination of the spectrum in the non-secular regime. Here, Eq. (37) reduces to

$$\hat{p}_\phi = \frac{\Theta}{2}\hat{\mathbf{R}}^2 = \frac{\Theta}{S^2}\hat{H} = (1/\bar{\omega})\hat{H} \tag{40}$$

after the use of (20), (25), (26). The non-zero values of the spectrum

$$p_\phi = \hbar \left\{m + (1/2)\right\} \tag{41}$$

of the angular momentum operator

$$\hat{p}_\phi = (\hbar/2)\left\{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger\right\} = \hbar \left\{\hat{a}^\dagger \hat{a} + (1/2)\right\} \tag{42}$$

are therefore those of positive half-integers. Unlike the secular regime, there is now nothing to eliminate the factor of one-half in the right-hand side of Eq. (42).

The peculiar nature of the commutators (22) assists certain equal-time commutators involving the angular momentum operator to retain the same values in the non-secular Chern-Simons theory as they have in the secular theory. For example,

$$\left[\hat{p}_\phi, \hat{R}_\imath\right] = i\hbar \epsilon_{\imath\jmath} \hat{R}_\jmath, \tag{43a}$$

$$\left[\hat{p}_\phi, \hat{P}_\imath\right] = i\hbar \epsilon_{\imath\jmath} \hat{P}_\jmath. \tag{43b}$$

In the non-secular theory, a determination of Eq. (43) involves the non-canonical commutators (22) and momentum (20). Both of these equations introduce factors of one-half into the algebra, which are then summed to the same result as is obtained in the secular theory. Equations (43) are equivalent to the values of the corresponding commutators in three-dimensional Cartesian space [19].

V. DETERMINATION OF THE GENERAL EIGENFUNCTION

In this section a general determination is made for the eigenfunction of a particle whose motion is defined by the generic Lagrangian (1). The theory is applicable to both the secular and non-secular regimes. Again, in what follows, the parameter Θ is considered to be specifically a positive quantity. Nothing is lost by making this specialisation.

The determination of the eigenfunction of a charged particle in a constant magnetic field, is a problem with a long pedigree. Dingle, in an extensive series of papers in the early 1950s, investigated the problem in depth by relating it to the magnetic properties of metals in various situations [20], [21]. In the course of these investigations he determined the eigenfunction for an electron in a constant magnetic field in terms of confluent hypergeometric functions. He appears to have been the first to have couched the solution explicitly in terms off these functions. His method will be adapted in the present paper; the system of a charged particle in a constant magnetic field is prototypic to that defined by the generic Lagrangian (1), the only difference being that the particle is not usually considered subjected to a trapping potential and the absence of the last term in (1) makes the problem easier to solve.

The problem of determining the eigenfunction in non-secular regime is complicated by the appearance of an unorthodox identification of the canonical momentum operator $\hat{\mathbf{P}}$ in terms of spatial derivatives. Normally, the replacement $\hat{\mathbf{P}} \to -i\hbar\boldsymbol{\nabla}$ is made routinely in quantum theory, and, save with the restriction of the gradient operator $\boldsymbol{\nabla}$ to two dimensions, one might expect the same in the present case if the eigenfunction Ψ were to be obtained by solving Schrödinger's equation. However, in the non-secular limit, the commutator (22a) between the conjugate co-ordinate operators is half the canonical value. This has a profound effect on the dynamics of the problem. It ensures that in the non-secular theory the usual linear momentum identification must be replaced by $\hat{\mathbf{P}} \to -i(\hbar/2)\boldsymbol{\nabla}$. The presence of the factor of two in a denominator associated with the differential form of the linear momentum operator propagates through to the quantisation of the classical angular momentum (32), and, therefore, the usual identification of the angular momentum operator must be replaced by $\hat{p}_\phi \to -i(\hbar/2)(\partial/\partial\phi)$.

One might consider that while the non-secular theory identification $\hat{\mathbf{P}} \to -i(\hbar/2)\boldsymbol{\nabla}$ is consistent with the commutator (22a) it is inconsistent with the commutator (22c) between the linear momentum components. However, Eq. (22c) is equivalent to the pair of equations (22a) and (20). Thus, no inconsistencies will arise if the commutator (22c) — and the commutator (22b) — are regarded not in isolation but only as consequences of the fundamental commutator between the canonical operators, expressed by Eq. (22a), and the non-independence of these operators, expressed by Eq. (20).

To keep the formalism general, the identifications

$$\hat{\mathbf{P}} \to -i(\hbar/\varsigma)\boldsymbol{\nabla}, \tag{44a}$$

$$\hat{p}_\phi \to -i(\hbar/\varsigma)(\partial/\partial\phi) \tag{44b}$$

will be adopted, in conjunction with the Hamiltonian operator

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2M} - \frac{\Theta}{2M}\hat{p}_\phi + \frac{M\Omega^2}{2}\rho^2. \tag{45}$$

Equation (45) is obtained from (4) and (37). The parameter ς takes on the values of 1, 2 in the secular and non-secular regimes respectively — *q.v.* Eqs. (49b), (50b) below. If the two-dimensional gradient operator

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial R_1}, \frac{\partial}{\partial R_2}\right) = \left(\left[\cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right], \left[\sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right]\right) \tag{46}$$

is written in terms of the polar co-ordinates (31) then a general Schrödinger equation

$$-\frac{\hbar^2}{2\varsigma^2 M}\left\{\rho^{-1}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Psi}{\partial\rho}\right] + \rho^{-2}\frac{\partial^2\Psi}{\partial\phi^2}\right\} + i\hbar\frac{\Theta}{2\varsigma M}\frac{\partial\Psi}{\partial\phi} + \left[\frac{M\Omega^2}{2}\rho^2 - E\right]\Psi = 0 \tag{47}$$

for the polar eigenfunction $\Psi(\rho,\phi)$ may be constructed from the Hamiltonian (45) and operator identifications (44). Note the appearance of terms in $\hbar/\varsigma$. Associated with the solution of Eq. (47) are a set of boundary conditions that ensure that the eigenfunction Ψ is consistent with the physical interpretations of quantum mechanics, together with a set of supplementary conditions necessary to select the secular and non-secular regimes of the theory. The specific form of the well-behaved Schrödinger equation in the non-secular limit will be derived later in this section.

The usual boundary conditions of

$$\Psi \text{ remains finite when } \rho = 0, \tag{48a}$$

$$\Psi \to 0 \text{ as } \rho \to \infty \tag{48b}$$

are chosen. These restrict Ψ to those functions that are continuous at the origin and vanish at distances which greatly exceed the dimensions of the system.

The first set of supplementary conditions states that

$$M \text{ remains finite,} \tag{49a}$$

$$\varsigma = 1. \tag{49b}$$

These ensure that the Lagrangian (1) is second-order in velocity, and assign the usual value of unity to the parameter ς. This defines the secular regime, where the momentum-operator identifications (44) are reduced to their usually-seen forms. By the same token, the supplementary conditions (49) specialise the general equation (47) and give it a familiar appearance: one closely related to that which might typically be seen in the Schrödinger equation for an electron in a magnetic field.

The second set of supplementary conditions

$$M\Omega \to \frac{\Theta}{2} \ \text{ as } \ M \to 0, \tag{50a}$$

$$\varsigma = 2. \tag{50b}$$

characterises the regime of the non-secular theory. The set consists of the non-secular limit (24), which for consistency is repeated in (50a), and the value of ς, given by Eq. (50a), appropriate to the non-standard commutators (22). The supplementary conditions (50) have the effects of eliminating the secular term from the Lagrangian (1), which becomes first-order in velocity, and, in the quantised theory, of switching off the commutator (5) and switching on the commutators (22). It should be noted that the boundary condition (48a) is adopted in the non-secular theory, despite the fact that the commutator (22b) implies that the operators $\hat{R}_1$ and $\hat{R}_2$ do not commute. The eigenfunction Ψ is always a function of classical spatial variables — here, the polar co-ordinates ρ, ϕ — and therefore must remain continuous at the classical origin.

The Hamiltonian (45) and angular momentum (44b) commute. Consequently, the eigenfunction $\Psi(\rho,\phi) = \chi(\rho)\psi(\phi)$ is separable in terms of functions $\chi(\rho)$, $\psi(\phi)$. In particular,

$$\psi(\phi) = \exp[i\varsigma(p_\phi/\hbar)\phi] = \exp[i\varsigma l\phi] \tag{51}$$

is an eigenfunction of the angular momentum operator (44b), where

$$l = p_\phi/\hbar \tag{52}$$

is the angular momentum eigenvalue in units of $\hbar$, associated with the eigenoperator equation

$$\hat{p}_\phi \psi(\phi) = -i\frac{\hbar}{\varsigma}\frac{\partial \psi(\phi)}{\partial \phi} = p_\phi \psi(\phi). \tag{53}$$

It is apparent — since the parameter ς can assume a value other than unity — that the eigenfunction Ψ of the Hamiltonian may be single-valued without the need to impose the condition that l must be an integer. If Ψ is to be single-valued then it is the product ςl whose non-zero value must be a positive or negative integer. The non-zero values of the angular-momentum eigenvalue (52), although an integer in the secular theory, when $\varsigma = 1$, need not be an integer in the non-secular theory, when $\varsigma = 2$. Nevertheless, the eigenfunction

$$\Psi(\rho\phi) = \exp[i\varsigma l\phi]\chi(\rho) \tag{54}$$

of the general Schrödinger equation (47) is single-valued in both the secular and non-secular theories.

The details of the derivation of the eigenfunction $\Psi(\rho, \phi)$ from Eq. (47) are given in Appendix A. The general result can be written as the sum

$$\Psi(\rho, \phi) = \Psi_1 + \Psi_2 \tag{55}$$

of the two functions

$$\Psi_1 = \frac{u}{\rho}\left[\frac{\varsigma M\Omega}{\hbar}\rho^2\right]^{(\lambda+1)/2} \exp(i\varsigma l\phi)\exp\left[-\frac{\varsigma M\Omega}{2\hbar}\rho^2\right] {}_1F_1\left(-n;\, 1+\lambda;\, \frac{\varsigma M\Omega}{\hbar}\rho^2\right), \tag{56a}$$

$$\Psi_2 = \frac{v}{\rho}\left[\frac{\varsigma M\Omega}{\hbar}\rho^2\right]^{(-\lambda+1)/2} \exp(i\varsigma l\phi)\exp\left[-\frac{\varsigma M\Omega}{2\hbar}\rho^2\right] {}_1F_1\left(-n-\lambda;\, 1-\lambda;\, \frac{\varsigma M\Omega}{\hbar}\rho^2\right). \tag{56b}$$

Equations (56) are expressed in manifestly dimensionless forms. They are made up of the following components: two dimensionless integration constants u, v as prefactors, consistent with eigenfunction normalisation

$$\int_0^{2\pi} d\phi \int_0^\infty d\rho\ \rho\Psi^*(\rho, \phi)\Psi(\rho, \phi) = 1; \tag{57}$$

two functions of ρ that diverge at the origin if the parameter λ is negative [positive] in Eq. (56a) [Eq. (56b)]; two identical phase factors, mediated by ςl; two identical Gaussian functions; and, as postfactors, two hypergeometric functions, mediated by the parameters λ and n, of the type known as Kummer's function

$$ {}_1F_1(\alpha;\beta;y) = 1 + \frac{\alpha}{\beta}y + \frac{\alpha(\alpha+1)}{2!\beta(\beta+1)}y^2 + \ \ldots \tag{58} $$

The parameter

$$ \lambda = \pm\varsigma l \tag{59} $$

is introduced into the formalism as a mathematical expediency, in order to reduce the general Schrödinger equation (47) to one which is analytically soluble. Again, the details are given in Appendix A. The parameter n depends on the eigenenergy

$$ E = \frac{\hbar\Omega}{\varsigma}\left\{2n + 1 + \lambda - \frac{\varsigma l\Theta}{2M\Omega}\right\}. \tag{60} $$

From Eq. (59) and the single-valueness of the eigenfunction, λ can only be zero or an integer. The eigenenergy (60) appears to diverge in the non-secular regime, incorporated in the second set of supplementary conditions (50). However, the reader is reminded of the comments made at the end of Section III. The non-secular limit must not be applied to the eigenenergy in isolation, but only to the Schrödinger equation as a whole. This will be done at the end of the present section.

If λ vanishes the two components (56a), (56b) are identical. It will be seen later in Section VI that Eqs. (56a), (56b) are mutually exclusive in the sense that they do not occur together for non-zero values of λ. At this stage, discussion will be limited to general remarks on how the divergent features of hypergeometric functions are circumvented by the eigenfunction selecting only a single component in Eq. (55). Should λ be a negative integer then the denominator of one of the terms of the hypergeometric series ${}_1F_1\left(-n; 1+\lambda; (\varsigma M\Omega/\hbar)\rho^2\right)$ associated with Eq. (56a) vanishes and the series as a whole diverges, unless n is a positive integer and numerically equal to λ. A similar situation occurs in the hypergeometric series

${}_1F_1\left(-n-\lambda; 1-\lambda; (\varsigma M\Omega/\hbar)\rho^2\right)$ associated with Eq. (56b), should λ be a positive integer. Here, to prevent this from happening, n needs to be a negative integer and numerically equal to $(\lambda-1)$. However, these situations do not occur because of the boundary condition (48a). For vanishing y, Kummer's function (58) collapses to unity. Therefore, equation (56a) diverges at the origin if λ is negative — a similar divergence occurs in (56b) if λ is positive. Consequently, if λ were positive then the eigenfunction $\Psi(\rho,\phi)$ could consist of only the component (56a); if λ were negative then the eigenfunction would be composed only of the component (56b). This will be examined in detail later.

The asymptotic properties of Kummer's function (58) may be found in the standard texts on analysis. In general, the goal in determining the asymptotic expansion of a function $f(y)$ is to obtain a series expansion in $1/y$ for the function, which is valid in the region of large y. There are several inter-related procedures to do this in the case of Kummer's function. Operational methods using algebraic manipulation of differential operators as if they were numerical constants may be used to write Kummer's function in terms of a contour integral. This integral is then evaluated to produce the required series expansion. Another method uses the identification of Kummer's function as a Barnes's integral involving gamma functions. In general, the asymptotic properties of (58) impose severe restrictions on the nature of the factors α, β if the function ${}_1F_1(\alpha;\beta;y)$ is not to diverge as $y\to\infty$. Thus, the boundary condition (48b) means that n must be zero or a positive integer if the eigenfunction Ψ was composed of Eq. (56a) only. On the other hand, $n+\lambda$ must have similar values if Ψ was composed of Eq. (56b) only. Obviously, both these conditions must be met if neither (56a) nor (56b) were absent from the eigenfunction.

In order to demonstrate that the general Schrödinger equation (47) remains well-behaved in the non-secular limit it will be useful to introduce the ς-dependent operators

$$\hat{A}_{\pm} = \frac{\sqrt{\varsigma}}{2}\exp\left[(\pm 1)^{\varsigma} i\phi\right]\left\{\left[\frac{M\Omega}{\hbar}\right]^{\frac{1}{2}}\rho + \frac{1}{\varsigma}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho} + (\pm 1)^{\varsigma}\frac{i}{\varsigma\rho}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\}, \tag{61a}$$

$$\hat{A}^{\dagger}_{\pm} = \frac{\sqrt{\varsigma}}{2}\exp\left[-(\pm 1)^{\varsigma} i\phi\right]\left\{\left[\frac{M\Omega}{\hbar}\right]^{\frac{1}{2}}\rho - \frac{1}{\varsigma}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho} + (\pm 1)^{\varsigma}\frac{i}{\varsigma\rho}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\}. \tag{61b}$$

Note the forms of the Hermitean conjugates. It may be verified from Eq. (61) that

$$\left[\hat{A}_{\alpha}, \hat{A}^{\dagger}_{\alpha'}\right] = \delta_{\alpha\alpha'}, \qquad\qquad \alpha, \alpha' = -, + \tag{62}$$

for $\varsigma = 1$, 2. The general Schrödinger equation (47) can therefore be re-cast into the form

$$\frac{\hbar}{\varsigma} \sum_{\alpha=-,+} \omega_{\alpha} \left\{ \hat{A}^{\dagger}_{\alpha} \hat{A}_{\alpha} + \frac{1}{2} \right\} \Psi - E\Psi = 0, \tag{63}$$

involving the Boson operators (61), where ω_{α} is given by Eq. (9). The eigenenergy is equivalent to (60) — strictly, of course, as it appears in Eq. (63), it should be expressed as a function

$$E = \frac{\hbar\Omega}{\varsigma} \left\{ N_{+} + N_{-} + 1 + \frac{\Theta}{2M\Omega}(N_{+} - N_{-}) \right\} \tag{64}$$

of the integer quantum-numbers $N_{\pm}$. The parameters $N_{\pm}$ are the eigenvalues of the number operators $\hat{A}^{\dagger}_{\pm}\hat{A}_{\pm}$; They are associated with the Fock states $|N_{\pm}\rangle$ generated by the operators (61) according to the usual ladder-rules, after the manner of (10). If Eqs. (60), (64) are compared then it is apparent that

$$N_{\pm} = n + (1/2)(\lambda \mp \varsigma l). \tag{65}$$

Equation (65) will appear again in Sections VI, VII — in specialised forms, corresponding to the secular $\varsigma = 1$ and non-secular $\varsigma = 2$ regimes.

In the secular regime $\varsigma = 1$, the Hamiltonian (63) represents two uncoupled oscillators of frequencies $\omega_{\pm}$, consistent with (7). Therefore, the $\varsigma = 1$ form of the operator $\hat{A}_{\pm}$ ($\hat{A}^{\dagger}_{\pm}$) equals $\hat{a}_{\pm}$ ($\hat{a}^{\dagger}_{\pm}$) to within an arbitrary phase. Similarly, a comparison of the $\varsigma = 1$ form of the eigenenergy (64) with (12) shows that the quantum numbers $N_{\pm}$ are identical to $n_{\pm}$. A discussion of the $\varsigma = 1$ form of $\hat{A}_{\pm}$ ($\hat{A}^{\dagger}_{\pm}$) will be delayed until Section VI, when its explicit form will be seen to arise naturally.

As emphasised at the end of Section III, the non-secular limit must be made at the level of the Schrödinger equation if the eigenenergy spectrum is not to diverge. Thus, to go over to the non-secular regime, it is necessary to re-write Eq. (63) as

$$
\begin{aligned}
&\frac{\hbar}{\varsigma}\frac{M\Omega S^2}{\Theta^2}\left\{1+\sum_{\alpha=-,+}\hat{A}^\dagger_\alpha\hat{A}_\alpha\right\}\Psi+\frac{\hbar S^2}{2\varsigma\Theta}\left\{\hat{A}^\dagger_+\hat{A}_+-\hat{A}^\dagger_-\hat{A}_-\right\}\Psi \\
&=\frac{MS^2}{\Theta^2}\frac{\hbar\Omega}{\varsigma}\left\{N_++N_-+1+\frac{\Theta}{2M\Omega}(N_+-N_-)\right\}\Psi \\
&=\frac{MS^2}{\Theta^2}\frac{\hbar\Omega}{\varsigma}\left\{2n+1+\lambda-\frac{\varsigma l\Theta}{2M\Omega}\right\}\Psi \qquad (66)
\end{aligned}
$$

where a re-scaling by the dimensionless number MS^2/Θ^2 has been affected throughout and the general ς-dependent eigenenergy is written explicitly from the equivalent spectra (64), (60). The non-secular limit, in the form of the second set of supplementary conditions (50), is now applied to the Schrödinger equation (66). The result is

$$
\hbar\bar{\omega}\left\{\hat{A}^\dagger\hat{A}+\frac{1}{2}\right\}\Psi-E\Psi=0, \qquad (67)
$$

after using Eq. (26), where the re-scaled eigenenergy may be expressed in two equivalent forms:

$$
E=\hbar\bar{\omega}\left(N_++\frac{1}{2}\right), \qquad (68a)
$$

$$
E=\frac{\hbar S^2}{2\Theta}\left\{2n+1+\lambda-2l\right\}. \qquad (68b)
$$

The operators

$$
\hat{A}=\frac{1}{\sqrt{2}}\exp\left[i\phi\right]\left\{\left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\rho+\frac{1}{2}\left[\frac{\Theta}{2\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho}+\frac{i}{2\rho}\left[\frac{\Theta}{2\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\}, \qquad (69a)
$$

$$
\hat{A}^\dagger=\frac{1}{\sqrt{2}}\exp\left[-i\phi\right]\left\{\left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\rho-\frac{1}{2}\left[\frac{\Theta}{2\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho}+\frac{i}{2\rho}\left[\frac{\Theta}{2\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\} \qquad (69b)
$$

are the non-secular forms of (61), obtained by putting $\varsigma=2$ and replacing $M\Omega$ by $\Theta/2$. The commutator

$$
\left[\hat{A},\hat{A}^\dagger\right]=1 \qquad (70)
$$

remains appropriate for Boson operators. If Eqs. (67), (68a) are compared to the non-secular Hamiltonian (25) and eigenenergy (29) then it is apparent that $\hat{A}$ ($\hat{A}^\dagger$) equals $\hat{a}$ ($\hat{a}^\dagger$) to within an arbitrary phase and $N_+=m$. The eigenenergy spectrum (68a) is well-behaved and does not diverge. It is also without the erroneous factor of 2 which is present in the finite portion of (30). Therefore, the *ad hoc* finite subtraction mentioned at the end of Section III is not required. Of course, similar conclusions apply to the equivalent form (68b) of the eigenenergy; this will be discussed in Section VII.

The physical interpretations of the operators (61) and their specialisations will be discussed in Sections VI and VII, when the individual cases of $\varsigma = 1$, 2 are considered. Here, remarks will be limited to the form of the normalised eigenfunction

$$\tilde{\Psi}(\rho,\phi) = [\{k(\varsigma-1)\}!\pi]^{-1/2}\left[\frac{\varsigma M\Omega}{\hbar}\right]^{[1+k(\varsigma-1)]/2} \times \rho^{k(\varsigma-1)}\exp[ik(\varsigma-1)\phi]\exp\left[-\frac{\varsigma M\Omega}{2\hbar}\rho^2\right] \tag{71}$$

which is annihilated by both of the destruction operators given in Eq. (61a). Formally, the product $\hat{A}_\pm\tilde{\Psi}$ vanishes for the values of $\varsigma = 1$, 2 and for any value of k. Indeed, the normalisation constant is determined from (57) and a standard integral in the form

$$\int_0^\infty dx\; x^{2\tau+1}\exp[-2ax^2] = (\tau!/2^{\tau+2})a^{-(\tau+1)}, \tag{72}$$

which is valid for any value of τ — including those that are complex. It will be noticed that the factor of two which multiplies $\hbar$ in the Gaussian component of (71) does not occur in the normalisation component. Here, k need only be regarded as being either zero or a positive integer. However, the reader is cautioned against supposing that this necessarily implies the existence of an infinitely degenerate ground-state in the non-secular regime $\varsigma = 2$. Although the $\varsigma = 2$ forms of (71) are eigenfunctions of the non-secular Hamiltonian and angular momentum for general values of k, they are *simultaneous* eigenfunctions only for single and specific values of k. Equation (53) implies that a k-dependent angular momentum eigenvalue

$$\tilde{p}_\phi = \hbar k(\varsigma-1)/\varsigma \tag{73}$$

can be associated with (71). Equation (73) vanishes in the secular regime $\varsigma = 1$, where (71) reduces to

$$\tilde{\Psi}(\rho,\phi) = \frac{1}{\sqrt{\pi}}\left[\frac{M\Omega}{\hbar}\right]^{1/2}\exp\left[-\frac{M\Omega}{2\hbar}\rho^2\right]. \tag{74}$$

On the other hand, in non-secular regime $\varsigma = 2$, the value of (73) is $\hbar k/2$. It is therefore apparent from Eq. (42) — with the replacement of $\hat{a}^\dagger\hat{a}$ by $\hat{A}^\dagger\hat{A}$ — that in the non-secular regime the parameter k can only have the single value of unity, a value that is consistent with the boundary condition (48a). Therefore, in the non-secular regime, Eq. (71) reduces to

$$\tilde{\Psi}(\rho,\phi) = \frac{1}{\sqrt{\pi}}\left[\frac{\Theta}{\hbar}\right]\rho\exp[i\phi]\exp\left[-\frac{\Theta}{2\hbar}\rho^2\right]. \tag{75}$$

It will be confirmed in Sections VI and VII that (74) and (75) are indeed the ground-state eigenfunctions corresponding to the secular and non-secular regimes respectively. The peculiar nature of the eigenfunctions (71) will be discussed again in Section VII; it will be shown that k may be identified as N_-, the integer spectrum absent from the non-secular eigenenergy.

VI. SECULAR THEORY ($\varsigma = 1$)

This section describes the $\varsigma = 1$ specialisation of the eigenfunction and eigenenergy. The choice of $\varsigma = 1$ selects the secular theory: the Lagrangian is second-order in velocity since it contains the secular term; the equal-time conjugate commutators have their common forms (5); and, from Eq. (52), the product

$$\varsigma l = l = p_\phi/\hbar \tag{76}$$

is identical to the angular momentum eigenvalue (in units of $\hbar$). This is the usual theory. Equation (76), and the requirement of a single-valued eigenfunction, expressed formally by Eq. (54), immediately restricts the non-zero values of $p_\phi/\hbar$ to positive and negative integers. This corresponds to the Boson characteristic of the angular momentum operator (38).

Putting $\varsigma = 1$, the components (56) of the eigenfunction (55) become

$$\Psi_1 = u\left[\frac{M\Omega}{\hbar}\right]^{(\lambda+1)/2}\rho^{\lambda}\exp(il\phi)\exp\left[-\frac{M\Omega}{2\hbar}\rho^2\right]{}_1F_1\left(-n;\,1+\lambda;\,\frac{M\Omega}{\hbar}\rho^2\right), \tag{77a}$$

$$\Psi_2 = v\left[\frac{M\Omega}{\hbar}\right]^{(-\lambda+1)/2}\rho^{-\lambda}\exp(il\phi)\exp\left[-\frac{M\Omega}{2\hbar}\rho^2\right]{}_1F_1\left(-n-\lambda;\,1-\lambda;\,\frac{M\Omega}{\hbar}\rho^2\right). \tag{77b}$$

Similarly, the secular theory eigenenergy

$$E = \hbar\Omega \left\{ 2n + 1 + \lambda - \frac{l\Theta}{2M\Omega} \right\} \tag{78}$$

is obtained from (60). The boundary conditions described in the previous section dictate that n must be zero or a positive integer. If l, and therefore λ, were to vanish then the components (77) would have to be identical — ignoring u, v. The choices of the signs of the non-zero values of λ and l are not independent: they must be made consistent with the requirement that the eigenfunction $\Psi(\rho, \phi)$ be finite at the origin $\rho = 0$. After noting that l can have positive and negative integer-values, corresponding to negative and positive angular momentum, Eq. (59) would seem to imply the existence of two further possibilities:—

1. The integer-values of λ are chosen to be positive. The eigenfunction therefore consists of the single component (77a), and the eigenenergy is (78) but with λ now expressly a positive integer.

2. The integer-values of λ are chosen to be negative. The eigenfunction therefore consists of the single component (77b), and the eigenenergy would appear to be (78) but with λ now expressly a negative integer.

These two possibilities are identical. In the case of the second possibility, if λ is written as $-|\lambda|$, the boundary conditions ensure that $n - |\lambda|$ must be a positive integer, m, say. Therefore, a substitution along these lines in Eq. (77b) gives an eigenfunction which is equivalent to (77a), with m and n playing identical rôles. Similarly, a substitution in Eq. (78) produces an eigenenergy $E = \hbar\Omega\{2m + 1 + |\lambda| - (l\Theta/2M\Omega)\}$. This is equivalent to the eigenenergy corresponding to the first possibility.

To summarise: although Eq. (77) implies that the two eigenfunctions solutions are in principle independent, the boundary conditions result in only a single eigenfunction

$$\begin{aligned} \Psi_{nl}(\rho, \phi) = w \left[\frac{M\Omega}{\hbar} \right]^{(|l|+1)/2} & \rho^{|l|} \exp(il\phi) \\ \times \exp\left[-\frac{M\Omega}{2\hbar}\rho^2 \right] & {}_1F_1 \left(-n; 1 + |l|; \frac{M\Omega}{\hbar}\rho^2 \right) \end{aligned} \tag{79}$$

in the secular theory ($\varsigma = 1$), where w is the normalisation factor. This eigenfunction is appropriate to the eigenenergy

$$E = \hbar\Omega \left\{ 2n + 1 + |l| - \frac{l\Theta}{2M\Omega} \right\}, \tag{80}$$

and is a function of two quantum numbers; one of which, n, can have only positive integer values, while the other, l, can assume both positive and negative integer values. It is evident from the eigenenergy spectrum (80) that changing the sign of l does not change the energy of the state providing that the sign of Θ is also changed. Eq. (80) gives the ground state eigenenergy as $\hbar\Omega$. This is identical to the value of $(\hbar/2)(\omega_- + \omega_+)$ implicit in the Hamiltonian (7).

In Eq. (79), the normalisation factor has been written as w and it may be evaluated from Eq. (57). This is easily done in the case of the ground-state eigenfunction

$$\Psi_{00}(\rho, \phi) = \frac{1}{\sqrt{\pi}} \left[\frac{M\Omega}{\hbar} \right]^{\frac{1}{2}} \exp\left[-\frac{M\Omega}{2\hbar}\rho^2 \right], \tag{81}$$

corresponding to vanishing n and l, where w is determined to be $1/\sqrt{\pi}$. The ground state eigenfunction is simply a Gaussian expression because the Kummer's function in (79) reduces to unity when n vanishes. Of course, Eq. (81) is identical to (74) — the specialisation of (71) corresponding to $\varsigma = 1$. Equation (81) will be used later in this section when the eigenfunction (79) will be re-expressed in its usual form, as a normalised function involving a Laguerre polynomial.

It might be questioned as to how the quantum numbers of the present section are related to the quantum numbers $n_\pm$ introduced in Section II — and therefore to $N_\pm$ introduced in Section V. The parameters n, l, λ arise from solving Schrödinger's equation. Initially they could be any complex numbers, with l and λ related through Eq. (59). However, their values are restricted by the introduction of the boundary conditions (48). On the other hand, the parameters $n_\pm$ are introduced immediately as positive integers — since they parameterise the Fock states generated from the vacuum by the action of the creation operators $\hat{a}_\pm^\dagger$. A comparison between the various quantum numbers may be made by considering the eigenenergies (12) and (78); with the understanding that while $n_\pm$ are taken to be positive integers no initial assumptions are made about the natures of λ, l and n. From these equations, one may make the identifications

$$n_- - n_+ = l, \tag{82a}$$

$$n_+ + n_- = 2n + \lambda. \tag{82b}$$

These imply

$$n_\pm = n + (1/2)(\lambda \mp l). \tag{83}$$

Since $n_\pm = N_\pm$, this corresponds to the $\varsigma = 1$ form of Eq. (65). If λ is chosen to be equal to $+l$ then

$$n_+ = n, \tag{84a}$$

$$n_- = n + l. \tag{84b}$$

Therefore, n, l must be integers since $n_\pm$ is an integer. Thus, either n and l are both positive, in which case λ is also a positive integer; or n is positive and l is negative, in which case λ is a negative integer but with $n > |\lambda|$. However, if λ is chosen to be equal to $-l$ then

$$n_+ = n - l, \tag{85a}$$

$$n_- = n. \tag{85b}$$

This again implies that n, l must be integers. And again, either n and l are positive, in which case λ is a negative integer but with $n > |\lambda|$ and $n > l$; or n is positive and l is negative, in which case λ is a positive integer.

Although the normalisation factor w in Eq. (79) may be determined for general n, l by the use of (57), it is instructive to generate normalised eigenfunctions from the ground state (81). This also allows the hypergeometric component of the eigenfunction to be replaced by an associated Laguerre polynomial. To begin, the Boson operators $\hat{a}_\pm$, $\hat{a}_\pm^\dagger$ are first re-cast in terms of polar variables (31). These operators were introduced through Eqs. (8); and by using

$$\hat{P}_1 = -i\hbar\cos\phi\frac{\partial}{\partial\rho} + \frac{i\hbar}{\rho}\sin\phi\frac{\partial}{\partial\phi}, \tag{86a}$$

$$\hat{P}_2 = -i\hbar\sin\phi\frac{\partial}{\partial\rho} - \frac{i\hbar}{\rho}\cos\phi\frac{\partial}{\partial\phi} \tag{86b}$$

they can be re-written as

$$\hat{a}_{\pm} = \frac{\exp(\pm i\phi)}{2}\left\{\left[\frac{M\Omega}{\hbar}\right]^{\frac{1}{2}}\rho + \left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho} \pm \frac{i}{\rho}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\}, \tag{87a}$$

$$\hat{a}^{\dagger}_{\pm} = \frac{\exp(\mp i\phi)}{2}\left\{\left[\frac{M\Omega}{\hbar}\right]^{\frac{1}{2}}\rho - \left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\rho} \pm \frac{i}{\rho}\left[\frac{M\Omega}{\hbar}\right]^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\right\}. \tag{87b}$$

Note the forms of the Hermitean conjugates. It will be seen that these operators are identical to the $\varsigma = 1$ specialisations of (61). It may be directly verified that (87) give the Boson equal-time commutator (11). Again, if Eqs. (87) are substituted into the Hamiltonian (7) then the correct polar-form of the Hamiltonian (47) is obtained, with $\varsigma = 1$. It may also be verified that the ground state (81) is annihilated by both $\hat{a}_-$ and $\hat{a}_+$. Equation (87) should be compared to (8). In the latter equation, the operators $\hat{a}_{\pm}$, $\hat{a}^{\dagger}_{\pm}$ are defined in terms of the Cartesian components of the quantum mechanical operators $\hat{\mathbf{R}}$, $\hat{\mathbf{P}}$. In the former equation, $\hat{a}_{\pm}$, $\hat{a}^{\dagger}_{\pm}$ are defined in terms of the classical variables ρ and ϕ. The occurrence of some of these classical variables as differential coefficients has the effect of making the functions defined by (87) behave as quantum mechanical operators, ensuring that (87) is consistent with the non-vanishing commutator (11).

The creation forms of the operators (87) allow the formation of two equivalent sets of normalised number states

$$|n_{\pm}, n_{\mp}\rangle = \frac{(\hat{a}^{\dagger}_{\pm})^{n_{\pm}}(\hat{a}^{\dagger}_{\mp})^{n_{\mp}}}{(n_{\pm}!)^{\frac{1}{2}}(n_{\mp}!)^{\frac{1}{2}}}, |0\rangle_{\pm}|0\rangle_{\mp} \tag{88}$$

and, subsequently, the eigenfunction

$$\Psi_{n_{\pm},n_{\mp}}(\rho,\phi) = \langle\rho,\phi|n_{\pm},n_{\mp}\rangle = \frac{(\hat{a}^{\dagger}_{\pm})^{n_{\pm}}(\hat{a}^{\dagger}_{\mp})^{n_{\mp}}}{(n_{\pm}!)^{\frac{1}{2}}(n_{\mp}!)^{\frac{1}{2}}}\left[\frac{M\Omega}{\pi\hbar}\right]^{\frac{1}{2}}\exp\left\{-\frac{M\Omega}{2\hbar}\rho^2\right\} \tag{89}$$

may be generated from the vacuum. The details of the calculation are given in Appendix B. The final result

$$\Psi_{n,l}(\rho,\phi) = \frac{(-1)^{n+|l|}}{\sqrt{\pi}} \frac{(n!)^{\frac{1}{2}}}{[(n+|l|)!]^{\frac{1}{2}}} \left[\frac{M\Omega}{\hbar}\right]^{(1+|l|)/2}$$
$$\times \rho^{|l|} \exp(il\phi) \exp\left\{-\frac{M\Omega}{2\hbar}\rho^2\right\} \mathcal{L}_n^{|l|} \left\{\frac{M\Omega}{\hbar}\rho^2\right\} \tag{90}$$

is written in terms of a principal quantum number n, and angular momentum quantum number l. Equation (90) is the polar form of eigenfunction of the Hamiltonian of the generic particle in the non-secular regime. It is expressed as a product of a Gaussian function and an associated Laguerre polynomial $\mathcal{L}_n^{|l|}$ in the squared radial distance ρ^2. The corresponding eigenenergy is given by Eq. (80). The reader's attention is drawn to Eq. (B13) in Appendix B, the definition of the associated Laguerre polynomial used in the present paper, and the accompanying comment.

VII. NON-SECULAR THEORY ($\varsigma = 2$)

This section describes the $\varsigma = 2$ specialisation of the eigenfunction and eigenenergy. The choice of $\varsigma = 2$ selects the non-secular theory: the Lagrangian is first-order in velocity since it does not contain the secular term, and the equal-time conjugate commutators have their non-canonical forms (22). The product

$$\varsigma l = 2l = 2p_\phi/\hbar = \text{zero or an integer} \tag{91}$$

is determined by Eq. (52). Formally, the product ςl is restricted to zero or integer values by the requirement that the eigenfunction be single-valued, a fact which is expressed by Eq. (54). However, a vanishing value does not occur for (91) since an examination of Eqs. (91) and (41) gives

$$l = \{m + (1/2)\}\,. \tag{92}$$

Thus, l is restricted to odd half-integer multiples of $\hbar$, corresponding to the Fermion characteristic of the angular momentum operator (42), with a minimum value of one-half.

Putting $\varsigma = 2$, the components (56) of the eigenfunction (55) become

$$\Psi_1 = u \left[\frac{\Theta}{\hbar}\right]^{(\lambda+1)/2} \rho^{\lambda} \exp(2il\phi) \exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(-n; 1+\lambda; \frac{\Theta}{\hbar}\rho^2\right), \tag{93a}$$

$$\Psi_2 = v \left[\frac{\Theta}{\hbar}\right]^{(-\lambda+1)/2} \rho^{-\lambda} \exp(2il\phi) \exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(-n-\lambda; 1-\lambda; \frac{\Theta}{\hbar}\rho^2\right) \tag{93b}$$

in the non-secular limit, where, from (A5),

$$\lambda = \pm 2l, \tag{94}$$

and u, v are normalisation constants. Again, the boundary condition (48b) dictates that either n or $n+\lambda$ must be zero or a positive integer. In the non-secular regime, neither the quantum number l nor the parameter λ can vanish. This is different from the secular theory described in Section VI. However, the choice of the sign of λ must be made consistent with the requirement that the eigenfunction $\Psi(\rho,\phi)$ be finite at the origin $\rho = 0$.

As in the secular regime, detailed in Section VI, the relationship between λ and l, which in the non-secular regime is given by Eq. (94), would seem to imply that (93a) and (93b) are independent. There appear to be two possibilities, associated with positive and negative values of λ. However, this is not the case. As in the secular theory, the two possibilities are identical — as may be demonstrated in a manner similar to that given in Section VI.

It was shown at the end of Section V that the non-secular eigenenergy in the form (68a) is well-behaved. A similar conclusion may be made for the form (68b). The sign of λ may again be chosen as positive without loss of generality, and it will be seen below that such a choice is consistent with a properly constituted eigenfunction in the form of an associated Laguerre polynomial. Therefore, Eq. (68b) gives

$$E = \frac{\hbar S^2}{\Theta}\left\{n + \frac{1}{2}\right\} \tag{95}$$

when $\lambda = 2l$ is used. This, together with (26), (29), (68a), (92) implies that

$$N_+ = n = m, \tag{96}$$

$$2l = 2n + 1. \tag{97}$$

The eigenfunction and eigenenergy may therefore be reduced to functions of a single quantum number — the principle quantum number n. Of course, the fact that $N_+ = n$ follows immediately from (65) with $\varsigma = 2$, $\lambda = 2l$.

To summarise: although Eq. (93) implies that the two eigenfunctions solutions are in principle independent, the boundary conditions result in the single eigenfunction

$$\begin{aligned}\Psi_n(\rho,\phi) = w\left[\frac{\Theta}{\hbar}\right]^{n+1} \rho^{2n+1} \exp\left[i(2n+1)\phi\right] \\ \times \exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(-n;\, 2+2n;\, \frac{\Theta}{\hbar}\rho^2\right)\end{aligned} \tag{98}$$

in the non-secular theory ($\varsigma = 2$) and appropriate to an eigenenergy (95) which is a function of n only. The ground state eigenenergy $\hbar S^2/2\Theta$ is identical to the value implicit in (29). The ground-state eigenfunction

$$\Psi_0(\rho,\phi) = \frac{1}{\sqrt{\pi}}\left[\frac{\Theta}{\hbar}\right] \rho \exp[i\phi] \exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] \tag{99}$$

in the non-secular regime is given by (98), with $n = 0$. The value of the normalisation factor w is obtained by the standard integral (72). Equation (99) is identical to (75), the specialisation of (71) corresponding to $\varsigma = 2$.

Rather than determine in the general case the normalisation constant w in Eq. (98) directly from (57) it is instructive, as in the previous section, to generate normalised eigenfunctions by means of the creation form of the operators (69). First, however, it is appropriate to make some general comments about these operators. It is problematic to think of them as representing the non-secular limiting forms of (87). Even with the replacement of $M\Omega$ by $\Theta/2$, Eq. (87) does not become identical to (69). Rather, the two sets of operators are special cases of (61), corresponding to $\varsigma = 1,\, 2$. This is consistent with the remarks made in Section III, to the effect that the single oscillator of the non-secular regime is a new oscillator and not one remaining from the secular regime. That said, it is certainly possible to understand why the operators (69) have their particular forms. The factor of two in the denominators of terms in $\partial/\partial\rho$ and $\partial/\partial\phi$ is to be expected from a ς-generalisation of Eq. (86), and is consistent with Eq. (44a). The prefactors of $1/\sqrt{2}$ in (69) ensure the canonical form of the commutator (70); they are the non-secular specialisations of the prefactors $\sqrt{\varsigma}/2$ in (61). They are not present in the non-secular Cartesian representations of $\hat{a}$ and $\hat{a}^\dagger$, given by (27). Such prefactors are necessary here in order to compensate for the fact that the variables in the polar representation (69)) are classical. Therefore, no non-vanishing commutators can be formed which are equivalent to (22b) and (22c); it is these particular commutators that must be used in a non-secular determination of (11) from Eq. (27).

In the non-secular case, in order to generate normalised eigenfunctions

$$\Psi_n(\rho,\phi) = \langle \rho,\phi|(n)\rangle = \frac{\left(\hat{A}^\dagger\right)^n}{(n!)^{1/2}}\tilde{\Psi}_k(\rho,\phi), \tag{100}$$

the creation operator (69b) has to act on the k-dependent function

$$\tilde{\Psi}_k(\rho,\phi) = \frac{1}{\sqrt{k!\pi}}\left[\frac{\Theta}{\hbar}\right]^{[1+k]/2}\rho^k \exp[ik\phi]\exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(0;1+k;\frac{\Theta}{\hbar}\rho^2\right). \tag{101}$$

The brackets in the ket $|(n)\rangle$ of Eq. (100) denote the presence of the angular momentum quantum number l, subsumed in n, through Eq. (97). This is a notational indication that the dimensions of the vectors $|(n)\rangle$ and $\langle\rho,\phi|$ are identical. Equation (101) is obtained by putting $\varsigma = 2$ in (71) and appending the appropriate unity-equivalent Kummer function as a post-factor. Since there exists a phase-term in ϕ in Eq. (101), the action

$$\hat{A}^\dagger\tilde{\Psi}_k = -\frac{k}{\sqrt{k!\pi}}\left[\frac{\Theta}{\hbar}\right]^{k/2}\rho^{k-1}\exp(i[k-1]\phi)\exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(-1;1+k-1;\frac{\Theta}{\hbar}\rho^2\right) \tag{102}$$

of the creation operator (69b) is to increase by minus-one the integer-arguments of the Kummer's function, as well as the exponents of ρ and the phase. Further applications of the creation operator lead to the equation

$$\begin{aligned}\left(\hat{A}^\dagger\right)^n\tilde{\Psi}_k = \frac{(-1)^n}{\sqrt{k!\pi}}&\left\{\prod_{p=0}^{n-1}(k-p)\right\}\left[\frac{\Theta}{\hbar}\right]^{[1+k-n]/2}\rho^{k-n}\exp(i[k-n]\phi)\\ &\times\exp\left[-\frac{\Theta}{2\hbar}\rho^2\right] {}_1F_1\left(-n;1+k-n;\frac{\Theta}{\hbar}\rho^2\right).\end{aligned} \tag{103}$$

In order to generate simultaneous eigenfunctions of the angular momentum and Hamiltonian the equation

$$k = 3n + 1 \tag{104}$$

must hold. This follows from (97) and (53), with $\varsigma = 2$. Since $\lambda = 2l$, a substitution of Eq. (97) into (65), again with $\varsigma = 2$, shows that the parameter

$$k = N_- \tag{105}$$

may be identified with that part of the integer spectrum that is present in the secular eigenenergy but absent in the non-secular eigenenergy. Therefore using (104), Eq. (103) becomes

$$\begin{aligned}\Psi_n(\rho,\phi) = \langle \rho,\phi|(n)\rangle = \frac{(-1)^n}{\sqrt{\pi}}[n!(3n+1)!]^{-\frac{1}{2}} \left\{\prod_{p=0}^{n-1}(3n+1-p)\right\}\left[\frac{\Theta}{\hbar}\right]^{n+1} \\ \times\, \rho^{2n+1}\exp[i(2n+1)\phi]\exp\left\{-\frac{\Theta}{2\hbar}\rho^2\right\}\, {}_1F_1\left(-n;\, 2+2n;\, \frac{\Theta}{\hbar}\rho^2\right).\end{aligned} \tag{106}$$

This is the normalised version of the eigenfunction (98). Finally, the identity (B11) in Appendix B leads to the polar form

$$\begin{aligned}\Psi_n(\rho,\phi) = \frac{(-1)^n}{\sqrt{\pi}}\frac{(n!)^{\frac{1}{2}}}{[(3n+1)!]^{\frac{1}{2}}}\left[\frac{\Theta}{\hbar}\right]^{n+1}\rho^{2n+1} \\ \times \exp\left[i(2n+1)\phi\right]\exp\left\{-\frac{\Theta}{2\hbar}\rho^2\right\}\mathcal{L}_n^{2n+1}\left\{\frac{\Theta}{\hbar}\rho^2\right\}\end{aligned} \tag{107}$$

of the eigenfunction in the non-secular theory written in terms of an associated Laguerre polynomial. As has already been mentioned, the single quantum number n determines both the eigenenergy (95) and the angular momentum — the latter by means of Eq. (97).

The secular theory eigenfunction (90) reduces to (107) by affecting the replacements

$$M \to \mu, \tag{108a}$$

$$\Omega \to \bar{\omega}, \tag{108b}$$

$$\hbar \to \hbar/2, \tag{108c}$$

$$n \to n, \tag{108d}$$

$$|l| \to 2n + 1, \tag{108e}$$

$$\pm\phi \to \phi - \pi, \tag{108f}$$

where μ is the dynamical mass (19) and $\bar{\omega}$ is the single oscillator frequency (26) of the non-secular theory. Together, Eqs. (108a) and (108b) give the non-secular limit (24), forming the supplementary condition (50a). Equation (108c) reflects the quantum manifestation of the classical constraint (16); an indication of the non-secular commutators (22) and supplementary condition (50b). The principal quantum number n of the secular theory is unchanged in the non-secular theory — a condition represented by Eq. (108d) — where it is now the sole quantum number, as represented by Eq. (108e). Equation (108f) is necessary because of the difference in phases between the secular and non-secular eigenfunctions. Equation (108f) is obtained by using (108e) to replace the factor $(-1)^{|l|}\exp(il\phi)$ by $\exp[i(2n+1)(\pi \pm \phi)]$. The $\pm$ sign in (108f) refers to the fact that the angular momentum can have both positive and negative eigenvalues in the secular regime.

VIII. EIGENFUNCTIONS IN CARTESIAN CO-ORDINATES

This section examines how the secular and non-secular eigenfunctions, determined in the previous sections, are related to the usual Hermite polynomial eigenfunctions of the harmonic oscillator.

To review the standard procedure [22], the secular regime eigenfunction in Cartesian co-ordinates is now determined. Two sets of Boson operators

$$\hat{b}_\imath = \left[\frac{M\Omega}{2\hbar}\right]^{\frac{1}{2}} \hat{R}_\imath + i\left[2M\Omega\hbar\right]^{-\frac{1}{2}} \hat{P}_\imath, \tag{109a}$$

$$\hat{b}_\imath^\dagger = \left[\frac{M\Omega}{2\hbar}\right]^{\frac{1}{2}} \hat{R}_\imath - i\left[2M\Omega\hbar\right]^{-\frac{1}{2}} \hat{P}_\imath \tag{109b}$$

are introduced; these are linear combinations

$$\hat{b}_1 = 2^{-\frac{1}{2}}\left(\hat{a}_- + \hat{a}_+\right), \tag{110a}$$

$$\hat{b}_2 = -i2^{-\frac{1}{2}}\left(\hat{a}_+ - \hat{a}_-\right) \tag{110b}$$

of the operators defined in (8). The differential equation

$$\left[\frac{M\Omega}{2\hbar}\right]^{\frac{1}{2}} R_\imath\langle R_\imath|\beta_\imath\rangle + \left[\frac{\hbar}{2M\Omega}\right]^{\frac{1}{2}} \nabla_\imath\langle R_\imath|\beta_\imath\rangle = \beta_\imath\langle R_\imath|\beta_\imath\rangle \tag{111}$$

may be constructed from coherent states

$$\hat{b}_\imath|\beta_\imath\rangle = \beta_\imath|\beta_\imath\rangle \tag{112}$$

of the annihilation operator (109a), and positional eigenstates

$$\hat{R}_\imath|R_\imath\rangle = R_\imath|R_\imath\rangle \tag{113}$$

of the operator $\hat{R}_\imath$. Use has been made of the secular regime ($\varsigma = 1$) specialisation of the identification (44a) in writing Eq. (111), whose solution

$$\langle R_\imath|\beta_\imath\rangle = \left\{\frac{M\Omega}{\pi\hbar}\right\}^{\frac{1}{4}} \exp\left\{-\frac{M\Omega}{2\hbar}R_\imath^2 + \left[\frac{2M\Omega}{\hbar}\right]^{\frac{1}{2}} \beta_\imath R_\imath - \frac{|\beta_\imath|^2}{2} - \frac{\beta_\imath^2}{2}\right\} \tag{114}$$

is normalised according to $\int_{-\infty}^{\infty} dR_\imath |\langle R_\imath|\beta_\imath\rangle|^2 = 1$. The coherent states

$$|\beta_\imath\rangle = \exp\left[-\frac{|\beta_\imath|^2}{2}\right] \exp(\beta_\imath \hat{b}_\imath^\dagger)|0\rangle_\imath \tag{115}$$

may be written in terms of two ground states $|0\rangle_1$, $|0\rangle_2$, enabling the expectation value

$$\langle R_\imath|\beta_\imath\rangle = \exp\left[-\frac{|\beta_\imath|^2}{2}\right] \sum_{m_\imath=0}^{\infty} \frac{\beta_\imath^{m_\imath}}{\sqrt{(m_\imath!)}} \langle R_\imath|m_\imath\rangle \tag{116}$$

to be expressed as an infinite series of expectation values involving number states $|m_\imath\rangle$. If now equation (114) is substituted into (116) one obtains

$$\langle R_\imath|m_\imath\rangle = \left[\frac{M\Omega}{\pi\hbar}\right]^{\frac{1}{4}} (2^{m_\imath} m_\imath!)^{-\frac{1}{2}} \exp\left(-\frac{M\Omega}{2\hbar}R_\imath^2\right) \mathcal{H}_{m_\imath}\left(R_\imath\sqrt{\frac{M\Omega}{\hbar}}\right) \tag{117}$$

after the use of a standard identity for an Hermite polynomial $\mathcal{H}_m$ of order m in the form

$$\exp\left\{\left[\frac{2M\Omega}{\hbar}\right]^{\frac{1}{2}} \beta_\imath R_\imath - \frac{\beta_\imath^2}{2}\right\} = \sum_{m=0}^{\infty} \left[2^{\frac{m}{2}} m!\right]^{-1} \beta_\imath^m \mathcal{H}_m\left[R_\imath\sqrt{\frac{M\Omega}{\hbar}}\right]. \tag{118}$$

There should be no confusion between the use here of the letter m, either in the form of $m_\imath$ or as a general index in Eq. (118), with the non-secular quantum number introduced in Eq. (29). Hermite polynomials obey the recursive formula

$$\mathcal{H}_{m+1}(y) = 2y\mathcal{H}_m(y) - 2m\mathcal{H}_{m-1}(y), \tag{119}$$

with $\mathcal{H}_0(y) = 1$ and $\mathcal{H}_{-1}(y)$ conventionally put equal to zero. The normalised eigenfunction

$$\begin{aligned}\Psi_{m_1,m_2}(R_1,R_2) &= \langle R_1|\langle R_2|m_1\rangle|m_2\rangle \\ &= \left[\frac{M\Omega}{2^{m_1+m_2}\pi\hbar m_1!m_2!}\right]^{\frac{1}{2}} \exp\left(-\frac{M\Omega}{2\hbar}\mathbf{R}^2\right) \\ &\times \mathcal{H}_{m_1}\left[R_1\sqrt{\frac{M\Omega}{\hbar}}\right]\mathcal{H}_{m_2}\left[R_2\sqrt{\frac{M\Omega}{\hbar}}\right] \end{aligned} \tag{120}$$

in the two-dimensional Cartesian space $\mathbf{R}$ of the secular regime is therefore determined to be proportional to the product of two Hermite polynomials, whose orders m_1, m_2 represent two sets of quantum numbers.

The secular eigenfunctions in Cartesian space — the Hermite polynomial form (120) — are equivalent to eigenfunctions in polar co-ordinates — the associated Laguerre polynomial form (90). One may be written as linear combinations of the other. For example,

$$\Psi_{0,0}(\rho,\phi) = \Psi_{0,0}(R_1,R_2), \tag{121a}$$

$$\Psi_{0,\pm1}(\rho,\phi) = \frac{-1}{\sqrt{2}}\left\{\Psi_{1,0}(R_1,R_2) \pm i\Psi_{0,1}(R_1,R_2)\right\}, \tag{121b}$$

$$\Psi_{0,\pm2}(\rho,\phi) = \frac{1}{\sqrt{2}}\left\{\Psi_{2,0}(R_1,R_2) - \Psi_{0,2}(R_1,R_2) \pm i\sqrt{2}\Psi_{0,1}(R_1,R_2)\right\}, \tag{121c}$$

$$\Psi_{1,0}(\rho,\phi) = \frac{1}{\sqrt{2}}\left\{\Psi_{2,0}(R_1,R_2) + \Psi_{0,2}(R_1,R_2)\right\} - \Psi_{0,0}(R_1,R_2). \tag{121d}$$

The normalisation procedure of the eigenfunctions in polar co-ordinates — the left-hand side of (121) — is given by Eq. (57). The procedure may be re-stated for the eigenfunctions in Cartesian co-ordinates — the right-hand side of (121) — as the integrations

$$\int d^2\mathbf{R}\ \Psi^*_{m_\imath,m_\jmath}(R_1,R_2)\Psi_{m_{\imath'},m_{\jmath'}}(R_1,R_2) = \delta_{m_\imath,m_{\imath'}}\delta_{m_\jmath,m_{\jmath'}}, \tag{122a}$$

$$\int d^2\mathbf{R}\ \Psi^*_{m_\imath,m_\jmath}(R_1,R_2)\Psi_{m_\jmath,m_\imath}(R_1,R_2) = 1 \tag{122b}$$

throughout all two-dimensional Cartesian space $\mathbf{R} = (R_1, R_2)$, where (122a) applies when $m_\imath \neq m_{\jmath'}$, $m_\jmath \neq m_{\imath'}$. The examples in (121) may be determined directly, using the recursive Eqs. (119), (B12) and the general relationship

$$x + iy = (1/2)[\mathcal{H}_1(x)\mathcal{H}_0(y) + i\mathcal{H}_0(x)\mathcal{H}_1(y)]. \tag{123}$$

Alternatively, one may use the standard identity

$$\sum_{k=0}^{n+m}(2i)^k P_k^{(n-k,m-k)}(0)\mathcal{H}_{n+m-k}(x)\mathcal{H}_k(y) = 2^{n+m}(-1)^m m! \\ \times (x+iy)^{n-m}\mathcal{L}_m^{n-m}(x^2+y^2), \tag{124a}$$

$$P_k^{(n-k,m-k)}(0) = \frac{(-1)^k}{2^k k!}\left[\frac{d^k}{dt^k}\{(1-t)^n(1+t)^m\}\right]_{t=0}, \tag{124b}$$

for $n \geq m$, between associated Laguerre polynomials and products of Hermite polynomials [23].

A technique similar to that used in obtaining Eq. (120) may be used to determine the eigenfunction in the two-dimensional Cartesian space of the non-secular regime. Here, only oscillator is present in the theory and the coherent states used are eigenstates of the pair of annihilation operators

$$\hat{\varpi}_1 = \hat{a} = \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\hat{R}_1 + i\left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\hat{P}_1, \tag{125a}$$

$$\hat{\varpi}_2 = i\hat{a} = \left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}}\hat{R}_2 + i\left[\frac{2}{\Theta\hbar}\right]^{\frac{1}{2}}\hat{P}_2. \tag{125b}$$

These are constructed by writing the non-secular annihilation operator (27a) in two equivalent forms by means of Eq. (20). The use of the non-secular regime ($\varsigma = 2$) specialisation of the identification (44a) allows the differential equation

$$\left[\frac{\Theta}{2\hbar}\right]^{\frac{1}{2}} R_\imath\langle R_\imath|\varpi_\imath\rangle + \left[\frac{\hbar}{2\Theta}\right]^{\frac{1}{2}}\nabla_\imath\langle R_\imath|\varpi_\imath\rangle = \varpi_\imath\langle R_\imath|\varpi_\imath\rangle \tag{126}$$

to be written down from coherent states

$$\hat{\varpi}_\imath|\varpi_\imath\rangle = \varpi_\imath|\varpi_\imath\rangle. \tag{127}$$

The eigenvalue equation $\hat{R}_\imath|R_\imath\rangle = R_\imath|R_\imath\rangle$ has also been used; although this equation is notionally the same as (113), the positional and momentum operators are not independent in the non-secular regime but are related through Eq. (20). Equation (126) is formally identical to (111), and a continuation of the same procedure as before leads to the normalised eigenfunction

$$
\begin{aligned}
\Psi_{m_1,m_2}(R_1,R_2) &= \langle R_1|m_1\rangle\langle R_2|m_2\rangle = \langle R_1|\langle R_2|m_1\rangle|m_2\rangle \\
&= \left[\frac{\Theta}{2^{m_1+m_2}\pi\hbar m_1!m_2!}\right]^{\frac{1}{2}} \exp\left(-\frac{\Theta}{2\hbar}\mathbf{R}^2\right) \\
&\times \mathcal{H}_{m_1}\left[R_1\sqrt{\frac{\Theta}{\hbar}}\right]\mathcal{H}_{m_2}\left[R_2\sqrt{\frac{\Theta}{\hbar}}\right]
\end{aligned} \tag{128}
$$

in the two-dimensional Cartesian space of the non-secular regime, written in terms of quantum numbers m_1, m_2. The secular theory eigenfunction (120) reduces to (128) by affecting the replacements (108a)–(108c), together with $m_\imath \to m_\imath$, for $\imath = 1, 2$.

The non-secular eigenfunctions in Cartesian space — the Hermite polynomial form (128) — are equivalent to eigenfunctions in polar co-ordinates — the associated Laguerre polynomial form (107). As in the secular theory, by using the standard identity (124), one eigenfunction may be written as linear combinations of the other. For example, for $n = 0$, 1,

$$
\Psi_0(\rho,\phi) = \frac{1}{\sqrt{2}}\left\{\Psi_{1,0}(R_1,R_2) + i\Psi_{0,1}(R_1,R_2)\right\}, \tag{129a}
$$

$$
\begin{aligned}
\Psi_1(\rho,\phi) = &\sqrt{\frac{5}{32}}\left\{\Psi_{5,0}(R_1,R_2) - i\Psi_{0,5}(R_1,R_2)\right\} - \frac{3}{\sqrt{32}}\left\{\Psi_{1,4}(R_1,R_2) - i\Psi_{4,1}(R_1,R_2)\right\} \\
&- \sqrt{\frac{2}{32}}\left\{\Psi_{3,2}(R_1,R_2) - i\Psi_{2,3}(R_1,R_2)\right\}.
\end{aligned} \tag{129b}
$$

Extending these relationships to further values of n draws on the Cartesian eigenfunction (128) in forms that rapidly involve higher and higher values of the quantum numbers m_1, m_2. Equations (129) are consistent with the normalisation procedure (122).

It has been shown in this section that the non-secular theory can be consistently expressed in Cartesian or polar co-ordinates. The latter representation is preferable for a transparent manifestation of angular momentum. An eigenfunction in polar co-ordinate may be expressed as a linear combination of Cartesian co-ordinate eigenfunctions; examples have been given in secular — Eq. (121) — and non-secular (129) regimes. Of course, one could equally reverse the formalism and write a Cartesian co-ordinate eigenfunction as a sum of polar co-ordinate eigenfunctions. Formally, all these transformations are similar to those used in expressing laser light, in the paraxial approximation, in terms of either

Laguerre-Gaussian or Hermite-Gaussian modes [24], [25].

IX. CONCLUSIONS AND DISCUSSION

The purpose of the present paper has been to determine the eigenfunctions and energy spectra for a non-relativistic particle moving simultaneously in an harmonic trapping potential and a gyroscopic potential. The determination has been made for the usual case where the Lagrangian contains the kinetic energy — the secular regime — and for the case where the kinetic energy is considered to be vanishingly small and therefore absent from the Lagrangian — the non-secular regime. It was thought that this might uncover a mechanism by which the exotic properties of the non-secular regime [1] could be experimentally observed.

The gyroscopic potential is proportional to the two-dimensional cross-product between the position and velocity vectors of the particle. It is indicative of a constraint. An ion in a constant magnetic field is considered to be the generic system under discussion, with the constant of proportionality Θ in the gyroscopic energy being the product of the magnetic field and the net ionic charge.

The eigenfunctions are determined by solving a ς-generalised Schrödinger equation (47) given in polar co-ordinates. The choices of $\varsigma = 1,\ 2$ select the secular and non-secular regimes respectively. The introduction of the parameter ς was dictated by the motivation of including in a single formalism the effects of dissimilar commutator relationships between the conjugate variables one encounters in the secular and non-secular regimes. It also ensures that the eigenfunctions remain single-valued in shifting from an angular momentum spectrum corresponding to integer multiples of $\hbar$ to one of half-integer multiples of $\hbar$.

Equations (55), (56) represent the solutions of the generalised Schrödinger equation. They involve ς-dependent products of power, Gaussian and hypergeometric functions in radial co-ordinates, and phase-factors in angular co-ordinates. The possible values of ς are itemised by two sets of supplementary conditions (49), (50), which, with the usual boundary conditions (48), are used to specialise the general solutions in Sections VI, VII. By the means of various mathematical identities, the hypergeometric functions are re-written as associated Laguerre polynomials. In this final form, the eigenfunctions in the secular and non-secular regimes are represented by Eqs. (90), (107) respectively. Similarly, the eigenenergies are given by Eqs. (80), (95); the latter equation, the eigenenergy in the non-secular regime, is well-behaved and does not diverge.

The well-behaved nature of the non-secular eigenenergy arises by applying the non-secular limit to the Schrödinger equation and not to its individual components. This was demonstrated by the introduction of ς-dependent Boson operators (61), allowing the general Schrödinger equation to be written as (66). The operators also permit normalisation with economy of effort, by generating normalised eigenstates from a base state. In the secular regime, these operators specialise to (87); to what one might have expected if one were to write the linear momentum in polar co-ordinates, as in (86). The normalised eigenstates are generated in (88) from a base corresponding to the vacuum. In the non-secular regime, the ς-dependent Boson operators specialise to forms which although not formally derivable from the secular case nevertheless have a recognisable structure. This was described in Section VII. The normalised eigenstates in the non-secular regime are generated in (100) from a base state (101). The base state (101) is parameterised (in terms of k) in order to be able to form normalised eigenfunctions of both the angular momentum and the Hamiltonian. The fact that they must be simultaneous eigenfunctions of these operators restricts the values of k to $3n + 1$. This also ensures that the vacuum state, corresponding to $n = 0$, $k = 1$ is not infinitely degenerate. The values of k were shown in Eq. (105) of Section VII to correspond to those of N_{-}, the integer spectrum that is present in the secular but not the non-secular eigenenergies.

The secular theory eigenfunction (90) reduces to the non-secular eigenfunction (107) in the non-secular limit defined by the replacements (108). The non-secular limit is complex: it is not simply the replacement of $M\Omega$ by $\Theta/2$, which is equivalent to Eqs. (108a), (108b) only. The peculiar nature of the commutator between the conjugate variables and the change in the angular momentum spectrum must be taken into account. These factors are accounted in Eqs. (108c), (108e).

In Section VIII, the eigenfunctions in Cartesian co-ordinates were determined by means of a standard technique involving the generation of coherent states as eigenstates of destruction operators. The eigenfunctions remain two-dimensional in the secular and non-secular regimes: they are given by Eqs. (120), (128) respectively, and involve products of Hermite polynomials. In the non-secular regime, the two-dimensionality of the eigenfunction is ensured by selecting the operator (125a) and another operator (125b) at right-angles, as eigenoperators of the coherent states. The eigenfunctions in polar co-ordinates can be expressed as linear combinations of the Cartesian eigenfunctions. This was demonstrated in (121) for the secular regime and in (129) for the non-secular regime. These transformations are well-known in optics, where they allow certain laser-light modes to be written in either Laguerre-Gaussian or Hermite-Gaussian forms. The relevance of these transformations may not be accidental. The Laguerre-Gaussian mode function has analogies in the formalism of an anisotropic oscillator [24]. It is therefore possible that there exists some regime of the paraxial approximation, perhaps involving an atomic interaction of the laser light, where the mode function has non-secular analogies.

The question was asked in Section I as to whether the non-secular regime was accessible physically by cooling the ion — the generic particle — sufficiently by means of the harmonic trapping potential, such that its kinetic energy is negligible compared to the other components in the secular Lagrangian (1). Such a vanishing kinetic energy would give rise to the non-secular Lagrangian (14). It has to be said that such direct accessibility appears to be unlikely. To pass from the secular-regime eigenfunction (90) to the eigenfunction (107) appropriate to non-secular theory requires the replacements (108). These equations seem to represent more than can be achieved by cooling — a physical process whose formal effects are propagated through the non-secular limit (24) to the substitutions (108a) and (108b).

If the non-secular theory were to be considered as representing only an approximation to the secular case then any experimental inaccessibility might be of little matter; especially since the full theory is exactly soluble. However, an inaccessibility of the non-secular regime to experiment raises the question of how the non-secular theory is to be interpreted. The non-secular eigenfunction is consistent with the tenets of non-relativistic quantum mechanics. It is a simultaneous eigenfunction of the Hamiltonian and angular momentum operators. It has been obtained by solving a valid and satisfactory Schrödinger equation. It is square-integrable and obeys the usual boundary conditions. One may therefore reasonably expect the eigenfunction to determine observables that are in principle observable. All of this might be explained away by arguing that a dynamical particle must by definition possess some kinetic energy. By ignoring it one cannot be surprised if the physics of the problem is altered. However, this argument gives to a classical quantity, the particle's kinetic energy, a significance that might seem odd in quantum mechanics.

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APPENDIX A: DETAILS OF THE SOLUTION OF THE GENERAL SCHRÖDINGER EQUATION

Details of the solution to the general Schrödinger equation (47) are presented in this appendix, where the procedure is based on the standard analysis of confluent hypergeometric and associated functions [26].

A substitution of Eq. (54) into (47) gives a second-order partial differential equation in χ, after dividing throughout by $\exp[i\varsigma l\phi]$. This differential equation can be reduced to the dimensionless form

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{4}\left[-1 + \frac{2\varsigma}{\hbar\Omega x}\left(E + \varsigma l\hbar\frac{\Theta}{2\varsigma M}\right) - \frac{(\varsigma^2 l^2 - 1)}{x^2}\right] y = 0 \tag{A1}$$

when the variables are re-scaled according to

$$y(\rho) = \rho\chi(\rho), \tag{A2a}$$

$$x(\rho) = (\varsigma M\Omega/\hbar)\rho^2. \tag{A2b}$$

In the course of the above calculation it is necessary to divide throughout by factors proportional to $\varsigma M\Omega/\hbar$. This is legitimate in the non-secular limit, where, providing Θ remains finite, $M\Omega \to \Theta/2$ does not vanish. The solution to Whittaker's form (A1) of the confluent hypergeometric equation may be achieved by changing

$$y = x^{(1+\lambda)/2}\exp(-x/2)z \tag{A3}$$

to a new variable z, where λ is some parameter. A substitution of (A3) into (A1) gives

$$x\frac{\partial^2 z}{\partial x^2} + (\lambda + 1 - x)\frac{\partial z}{\partial x} + \left[\frac{\varsigma}{2\hbar\Omega}\left(E + \varsigma l\hbar\frac{\Theta}{2\varsigma M}\right) - \frac{(\lambda+1)}{2} + \frac{(\lambda^2 - \varsigma^2 l^2)}{4x}\right] z = 0. \tag{A4}$$

The term in $(\lambda^2 - \varsigma^2 l^2)$ may be removed by the restriction

$$\lambda^2 = \varsigma^2 l^2. \tag{A5}$$

If this is done then Eq. (A4) is a form

$$x\frac{\partial^2 z}{\partial x^2} + (\lambda + 1 - x)\frac{\partial z}{\partial x} + nz = 0 \tag{A6}$$

of Kummer's equation, where

$$n = \frac{\varsigma}{2\hbar\Omega}\left(E + \varsigma l\hbar\frac{\Theta}{2\varsigma M}\right) - \frac{(\lambda+1)}{2} \tag{A7}$$

is a new parameter. A re-arrangement of (A7) gives the eigenenergy spectrum (60).

A series solution in positive powers of x may be proposed for the partial second-order differential equation (A6). If the lowest power occurring in the series is k then the solution will have the general form

$$z = \sum_j a_j x^{k+j}, \tag{A8}$$

where j is a non-negative index. The substitution of (A8) into (A6) gives an equation which must hold for every value of x. Equating coefficients in the case of the lowest power of x, which occurs when $j = 0$, gives the indicial equation

$$k^2 + \lambda k = 0. \tag{A9}$$

The two solutions of (A9) correspond to the recurrence relationships

$$a_{j+1} = \frac{(j-n)}{(j+1)(j+\lambda+1)}a_j, \qquad k = 0; \tag{A10a}$$

$$a_{j+1} = \frac{(j-\lambda-n)}{(j+1)(j-\lambda+1)}a_j, \qquad k = -\lambda. \tag{A10b}$$

It is apparent that (A10b) may be transformed into (A10a) by means of the replacements $n \to n + \lambda$ and $\lambda \to -\lambda$. Consequently, each solution of (A6) may be written in terms of Kummer's functions (58). This leads to the general solution (55), (56) of the Schrödinger equation (47).

APPENDIX B: SECULAR THEORY CALCULATION ($\varsigma = 1$)

It will be useful to simply the notation by writing $\gamma = \sqrt{M\Omega/\hbar}$. The operators (87) become

$$\hat{a}_\pm = \frac{\exp(\pm i\phi)}{2}\left\{\gamma\hat{\rho} + \frac{1}{\gamma}\frac{\partial}{\partial\rho} \pm \frac{i}{\gamma\hat{\rho}}\frac{\partial}{\partial\phi}\right\}, \tag{B1a}$$

$$\hat{a}^\dagger_\pm = \frac{\exp(\mp i\phi)}{2}\left\{\gamma\hat{\rho} - \frac{1}{\gamma}\frac{\partial}{\partial\rho} \pm \frac{i}{\gamma\hat{\rho}}\frac{\partial}{\partial\phi}\right\}, \tag{B1b}$$

and Eq. (89) becomes

$$\begin{aligned}\Psi_{n_\pm,n_\mp}(\rho,\phi) &= \frac{\exp(\mp i n_\pm\phi)}{2^{n_\pm}(n_\pm!)^{\frac{1}{2}}}\left\{\gamma\hat{\rho} - \frac{1}{\gamma}\frac{\partial}{\partial\rho} \pm \frac{i}{\gamma\hat{\rho}}\frac{\partial}{\partial\phi}\right\}^{n_\pm} \\ &\times \frac{\exp(\pm i n_\mp\phi)}{2^{n_\mp}(n_\mp!)^{\frac{1}{2}}}\left\{\gamma\hat{\rho} - \frac{1}{\gamma}\frac{\partial}{\partial\rho} \mp \frac{i}{\gamma\hat{\rho}}\frac{\partial}{\partial\phi}\right\}^{n_\mp} \frac{\gamma}{\pi^{\frac{1}{2}}}\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\}. \end{aligned}\tag{B2}$$

The ground state $\Psi_{0,0}$ picks up an exponential term in ϕ when multiplied

$$(\hat{a}^\dagger_\mp)^{n_\mp}\Psi_{00} = \frac{\gamma}{\pi^{\frac{1}{2}}}(\gamma\rho)^{n_\mp}\exp(\pm i n_\mp\phi)\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\} \tag{B3}$$

by the first operator $(\hat{a}^\dagger_\mp)^{n_\mp}$. Note the difference in sign of the ϕ-dependent exponential in (B3). This is caused by the difference in sign of the $\partial/\partial\phi$ term in the operators (B1b). Also through the action of the component term $\pm(i/\gamma\hat{\rho})(\partial/\partial\phi)$, it will be seen that the second operator $(\hat{a}^\dagger_\pm)^{n_+}$ ($(\hat{a}^\dagger_\pm)^{n_-}$) decreases (increases) by n_+ (n_-) integer-steps the argument of the ϕ-dependent exponential. The second operator is responsible too for the formation of the hypergeometric portion of the eigenfunction. This is achieved in two stages. Firstly, the hypergeometric function ${}_1F_1(1, n_\mp, \gamma^2\rho^2)$ is induced from a single multiplication

$$\begin{aligned}\hat{a}^\dagger_\pm(\hat{a}^\dagger_\mp)^{n_\mp}\Psi_{0,0} &= -n_\mp\frac{\gamma}{\pi^{\frac{1}{2}}}(\gamma\rho)^{n_\mp-1}\exp(\pm i[n_\mp-1]\phi)\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\}\left\{1-\frac{\gamma^2\rho^2}{n_\mp}\right\} \\ &= -n_\mp\frac{\gamma}{\pi^{\frac{1}{2}}}(\gamma\rho)^{n_\mp-1}\exp(\pm i[n_\mp-1]\phi)\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\}{}_1F_1(-1;n_\mp;\gamma^2\rho^2)\end{aligned} \tag{B4}$$

by $\hat{a}^\dagger_\pm$. Subsequent multiplications by $\hat{a}^\dagger_\pm$ step down the integer-arguments of the hypergeometric function by means of the identity

$${}_1F_1(\alpha-1;\beta-1;y) = \left\{\frac{\beta-1-y}{\beta-1} + \frac{y}{\beta-1}\frac{d}{dy}\right\}{}_1F_1(\alpha;\beta;y) \tag{B5}$$

[26], to give

$$\begin{aligned}(\hat{a}^\dagger_\pm)^{n_\pm}(\hat{a}^\dagger_\mp)^{n_\mp}\Psi_{0,0} &= (-1)^{n_\pm}\frac{\gamma}{\pi^{\frac{1}{2}}}\left\{\prod_{p=0}^{n_\pm-1}(n_\mp-p)\right\}(\gamma\rho)^{n_\mp-n_\pm} \\ &\times\exp(\pm i[n_\mp-n_\pm]\phi)\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\}{}_1F_1(-n_\pm;1+n_\mp-n_\pm;\gamma^2\rho^2) \\ &= (\hat{a}^\dagger_\mp)^{n_\mp}(\hat{a}^\dagger_\pm)^{n_\pm}\Psi_{00}.\end{aligned} \tag{B6}$$

The last line of (B6) comes about through using the identity

$$(-y)^m \left\{ \prod_{p=0}^{n-1} (m-p) \right\} {}_1F_1(-n; 1+m-n; y)$$
$$= (-y)^n \left\{ \prod_{p'=0}^{m-1} (n-p') \right\} {}_1F_1(-m; 1+n-m; y), \tag{B7}$$

and reflects the fact that $\left[(\hat{a}_\pm^\dagger)^{n_\pm}, (\hat{a}_\mp^\dagger)^{n_\mp}\right] = 0$. In the present context, equation (B7) may be written as

$$(-1)^{n_\pm} (\gamma\rho)^{2n_\mp} \left\{ \prod_{p=0}^{m_\pm - 1} (n_\mp - p) \right\} {}_1F_1(-m_\pm; 1+n_\mp - n_\pm; \gamma^2\rho^2)$$
$$= (-1)^{n_\mp} (\gamma\rho)^{2n_\pm} \left\{ \prod_{p'=0}^{m_\mp - 1} (n_\pm - p') \right\} {}_1F_1(-m_\mp; 1+n_\pm - n_\mp; \gamma^2\rho^2), \tag{B8}$$

after using $(-1)^n = (-1)^{-n}$ for n equal to an integer or zero. A substitution of (B6) into (B2) yields

$$\Psi_{n_\pm, n_\mp}(\rho\phi) = \Psi_{n_\mp, n_\pm}(\rho\phi) = (-1)^{n_\pm} \frac{\gamma}{\pi^{\frac{1}{2}}} (n_\pm!)^{-\frac{1}{2}} (n_\mp!)^{-\frac{1}{2}} \left\{ \prod_{p=0}^{n_\pm - 1} (n_\mp - p) \right\} (\gamma\rho)^{n_\mp - n_\pm}$$
$$\times \exp(\pm i[n_\mp - n_\pm]\phi) \exp\left\{ -\frac{\gamma^2\rho^2}{2} \right\} {}_1F_1(-n_\pm; 1+n_\mp - n_\pm; \gamma^2\rho^2). \tag{B9}$$

This leads immediately to

$$\Psi_{n_\pm, n_\mp}(\rho\phi) = \Psi_{n_\mp n_\pm}(\rho\phi) = (-1)^{n_\pm} \frac{\gamma}{\pi^{\frac{1}{2}}} \frac{(n_\pm!)^{\frac{1}{2}}}{(n_\mp!)^{\frac{1}{2}}} (\gamma\rho)^{n_\mp - n_\pm}$$
$$\times \exp(\pm i[n_\mp - n_\pm]\phi) \exp\left\{ -\frac{\gamma^2\rho^2}{2} \right\} \mathcal{L}_{n_\pm}^{n_\mp - n_\pm}(\gamma^2\rho^2) \tag{B10}$$

after using the relationship

$$\frac{1}{n!} \left\{ \prod_{p=0}^{n-1} (m-p) \right\} {}_1F_1(-n; 1+m-n; y) = \mathcal{L}_n^{m-n}(y) \tag{B11}$$

between Kummer's functions and associated Laguerre polynomials. The value of an associated Laguerre polynomial is conveniently determined from the recursive formula

$$y\mathcal{L}_n^{m+1}(y) = (n+m)\mathcal{L}_{n-1}^m(y) - (n-y)\mathcal{L}_n^m(y), \tag{B12}$$

with $\mathcal{L}_n^0(y) = \mathcal{L}_n(y)$ and $\mathcal{L}_0(y) = 1$. The reader is reminded that there exists several mutually incompatible definitions of an associated Laguerre polynomial. The one used in the present paper

$$\mathcal{L}_n^m(y) = \frac{1}{n!}\exp(y)y^{-m}\frac{d^n}{dy^n}\left\{\exp(-y)y^{n+m}\right\} \tag{B13}$$

may be found in Gradshteyn and Ryzhik's *Table of Integrals, etc.* [27].

The quantum numbers $n_\pm$ have been identified with n, l, λ in Section VI by comparing Eqs. (12) and (78). The requirement that $\lambda^2 = l^2$ leads to the following possibilities

1. $\lambda = l$. This corresponds to $l > 0$ if λ is regarded as beings always a positive number.

   (a) $n_\pm = n$ and $n_\mp = n+\lambda$. The associated Laguerre polynomial in (B10) is therefore $\mathcal{L}_n^\lambda$ and the factor $[n_\pm!/n_\mp!]^{1/2}$ is $[n!/(n+\lambda)!]^{1/2}$.

   (b) $n_\pm = n+\lambda$ and $n_\mp = n$. This choice must be rejected since it would lead to an associated Laguerre polynomial indexed with a negative number.

2. $\lambda = -l$ This corresponds to $l < 0$ if λ is regarded as beings always a positive number.

   (a) $n_\pm = n$ and $n_\mp = n+\lambda$. This choice is invalid for the same reasons as in 2(b).

   (b) $n_\pm = n+\lambda$ and $n_\mp = n$. The associated Laguerre polynomial in (B10) is therefore $\mathcal{L}_n^\lambda$ and the factor $[n_\pm!/n_\mp!]^{1/2}$ is $[n!/(n+\lambda)!]^{1/2}$.

The eigenfunction may therefore be written as

$$\begin{aligned}\langle\rho,\phi|n,n+\lambda\rangle = (-1)^{n+\lambda}\frac{\gamma}{\pi^{\frac{1}{2}}}\frac{(n!)^{\frac{1}{2}}}{[(n+\lambda)!]^{\frac{1}{2}}}(\gamma\rho)^\lambda \\ \times\exp(\pm i\lambda\phi)\exp\left\{-\frac{\gamma^2\rho^2}{2}\right\}\mathcal{L}_n^\lambda(\gamma^2\rho^2).\end{aligned} \tag{B14}$$

Equation (B14) translates to (90) after the substitution $\gamma = \sqrt{M\Omega/\hbar}$ is made, and after changing λ to $|l|$.

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