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*Journal of Physics B*, **33**, 4177 (2000)

Front Cover

J. Phys. B: At. Mol. Opt. Phys. **33** (2000) 4177–4191. Printed in the UK PII: S0953-4075(00)51161-4

# Coherence length for a trapped Bose gas

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Received 26 June 2000, in final form 14 July 2000

**Abstract.** Matter waves can be assigned a coherence length analogous to the corresponding quantity in optics. We show that there is a dramatic increase in coherence length associated with the appearance of Bose–Einstein condensation in a trapped ideal Bose gas. The large increase in coherence length occurs even for temperatures just below the critical temperature at which only a small proportion of the trapped atoms is to be found in the ground state.

## 1. Introduction

The observation of interference between atoms prepared in Bose–Einstein condensates [1–3] provided a remarkable demonstration of the wave nature of matter at the macroscopic level. In these experiments, the condensates appeared to behave as classical waves rather than as ensembles of point particles. The visibility of the interference fringes observed is determined, as in optics, by the coherence properties of the interfering atoms. The coherence properties are most readily and completely described in terms of coherence functions [4–6]. Indeed, the existence of coherence is a necessary requirement for interference, with the coherence length furnishing a critical measure. In this paper we introduce the coherence length for a trapped Bose gas as a measure of the distance over which interference can be observed. We show that condensation is accompanied by a dramatic increase in the coherence length. At very low temperatures, the coherence length approaches the size of the condensate as demonstrated in [7].

Bose–Einstein condensation is associated with the appearance of a macroscopic population of atoms in the motional ground state. The wavefunction for this ground state then plays a role analogous to the amplitude of a coherent field for the condensate [8–11]. Numerical simulations based on the ground-state wavefunction are in very good agreement with the results of experiment [12, 13].

Optical interference is conventionally associated with a well defined phase relationship between the overlapping fields. The mechanism that determines the phase for a condensate has received considerable attention ([14] and references therein). It is important to note, however, that the observation of interference fringes does not necessarily imply the existence of a predetermined relative phase between the condensates. Indeed, an interference pattern will appear even for two condensates, each with a precisely determined number of atoms and hence with no preferred phase [15–22].

It has recently been demonstrated that interference fringes occur when two spatially separated parts of the same condensate are allowed to overlap [3]. This experiment also showed that fringes could be observed at temperatures just below the critical temperature for condensate formation. At these temperatures, most of the trapped atoms are not in the

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condensed motional ground state and the visibility is reduced compared with that which is observed at lower temperatures. The distance between the separated parts of the condensate for which interference fringes occur appears to increase significantly as the temperature is reduced below its critical value. This is consistent with the sudden increase in coherence length described in this paper.

## 2. Optical coherence

The theories of classical and quantum-optical coherence are both well developed [23, 24]. It may be helpful, therefore, to start with a brief discussion of the role of coherence and coherence length in optical interferometry.

Consider an optical interferometer of the Mach–Zehnder or Michelson type. The difference between the path lengths will impose phase shifts on the waves travelling through the interferometer, leading to interference. The interference will only be observable if the difference in the path lengths is sufficiently small that random phase fluctuations in the optical source have not caused a loss of coherence between the two recombined waves. The visibility is determined by the first-order coherence function

$$G^{(1)}(\tau) = \langle E^*(t)E(t+\tau)\rangle \tag{2.1}$$

where $E$ is the complex electric field amplitude and $\tau$ is the time delay associated with the difference in the interferometer path lengths. The angled brackets denote a temporal average over $t$. The modulus of the complex coherence function is peaked at $\tau = 0$ and decays to zero for large values of $\tau$. The time delay characteristic of the decay of the coherence function is the coherence time, which may be written as [23]

$$(\Delta\tau)^2 = \frac{\int_{-\infty}^{\infty} \mathrm{d}\tau\, \tau^2 |G^{(1)}(\tau)|^2}{\int_{-\infty}^{\infty} \mathrm{d}\tau\, |G^{(1)}(\tau)|^2}. \tag{2.2}$$

It is often more natural to deal with coherence length rather than coherence time. In optics, the coherence length $\mathcal{L}$ is simply related to the coherence time by $\mathcal{L} = c\Delta\tau$ and represents the difference in the lengths of the interferometer arms for which the visibility is significantly reduced. For matter there is no correspondingly simple relationship between coherence length and coherence time, however, the first-order coherence function and the associated coherence length can be defined for any wave.

## 3. Coherence length for a single trapped particle

The aim of this paper is to determine the coherence length for trapped quantum particles. We begin by considering a single particle. In a single-particle interferometer, interference occurs between the probability amplitudes associated with different positions of the particle.

Consider a particle prepared in a motional pure state with wavefunction $\psi(\boldsymbol{r})$. The coherence between the state at two positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be measured by interference. The interference pattern takes the form of a probability density

$$p = |a\psi(\boldsymbol{r}_1) + b\psi(\boldsymbol{r}_2)|^2 \tag{3.1}$$

where $a$ and $b$ are geometrical factors. The visibility will depend on $\psi^*(\boldsymbol{r}_1)\,\psi(\boldsymbol{r}_2)$, which is the analogue of the optical coherence function (2.1). If the particle is prepared in a mixture of states $\psi_n(\boldsymbol{r})$ with probabilities $\rho_n$ then the coherence function becomes

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_n \rho_n \psi_n^*(\boldsymbol{r}_1)\psi_n(\boldsymbol{r}_2) = \langle \boldsymbol{r}_1|\hat{\rho}|\boldsymbol{r}_2\rangle \tag{3.2}$$

where $\hat{\rho}$ is the density operator and $|\boldsymbol{r}\rangle$ is a three-dimensional position eigenstate. This is closely related to the spatial correlation function introduced to measure the temperature of ultra-cold atoms [25]. Interference associated with the motional state of a single trapped particle has been demonstrated in ion traps [26–29].

The coherence function (3.2) will depend, in general, on both the distance and direction between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. A measure of the distance in the $x$-direction over which coherent phenomena can be observed is the coherence length $\mathcal{L}$ in the $x$-direction, which is defined by

$$\mathcal{L}^2 = \frac{\int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2\, |\rho(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 (x_1 - x_2)^2}{2\int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2\, |\rho(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2}. \tag{3.3}$$

A factor of two has been introduced in comparison with the optical coherence time (2.2) so that the coherence length for a pure state coincides with its width given by the uncertainty $\Delta x$. We note that the coherence length can be expressed in terms of the density and position operators [30],

$$\mathcal{L}^2 = \frac{\mathrm{Tr}(\hat{\rho}^2 \hat{x}^2) - \mathrm{Tr}(\hat{\rho}\hat{x}\hat{\rho}\hat{x})}{\mathrm{Tr}(\hat{\rho}^2)}. \tag{3.4}$$

The difference between the width of the motional wavepacket and its coherence length is best illustrated by considering the example of a thermal wavepacket [30]. In this state, the probability of finding the particle in the $n$th motional eigenstate of the trap is given by the Boltzmann formula

$$\rho_n = [1 - \exp(-\beta\hbar\omega)] \exp(-\beta n\hbar\omega) \tag{3.5}$$

where $\omega$ is the trap frequency (in the $x$-direction) and $\beta = (k_{\mathrm{B}}T)^{-1}$ is proportional to the inverse temperature. The width of the thermal wavepacket exceeds that for the zero-temperature ground state,

$$\Delta x^2 = \frac{\hbar}{2m\omega} \coth(\beta\hbar\omega/2) = \Delta x_0^2 \coth(\beta\hbar\omega/2). \tag{3.6}$$

In contrast, the coherence length for the thermal wavepacket is less than that for the zero-temperature ground state,

$$\mathcal{L}^2 = \frac{\hbar}{2m\omega} \tanh(\beta\hbar\omega/2) = \mathcal{L}_0^2 \tanh(\beta\hbar\omega/2). \tag{3.7}$$

This means that the thermal wavepacket has a greater spatial extent than the ground state wavepacket, but that the distance over which interference can occur is less than that for the ground state. The width of the thermal wavepacket is a monotonically increasing function of temperature, but its coherence length decreases monotonically with temperature.

## 4. Coherence length for a trapped Bose gas

In order to treat the coherence properties of a Bose gas of cold atoms, we require a second-quantized description [11, 31]. Hence we introduce the atom annihilation and creation operators $\hat{\psi}(\boldsymbol{r})$ and $\hat{\psi}^\dagger(\boldsymbol{r}')$. We will be concerned in this paper only with operators evaluated at equal times and so suppress all time dependences. The operators satisfy the bosonic commutation relations

$$[\hat{\psi}(\boldsymbol{r}), \hat{\psi}^\dagger(\boldsymbol{r}')] = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{4.1}$$

$$[\hat{\psi}(\boldsymbol{r}), \hat{\psi}(\boldsymbol{r}')] = [\hat{\psi}^\dagger(\boldsymbol{r}), \hat{\psi}^\dagger(\boldsymbol{r}')] = 0. \tag{4.2}$$

We can represent the observable properties of our Bose gas in terms of these operators. In particular, the density of atoms at position $\boldsymbol{r}$ is associated with the operator $\hat{\psi}^{\dagger}(\boldsymbol{r})\,\hat{\psi}(\boldsymbol{r})$ and the total atom number operator is $\int \mathrm{d}\boldsymbol{r}\,\hat{\psi}^{\dagger}(\boldsymbol{r})\,\hat{\psi}(\boldsymbol{r})$.

In an interference experiment on a trapped gas, we might measure the interference between matter waves associated with two different positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the gas [3]. The mean density of atoms would then be

$$n = \left\langle \left(a^{*}\hat{\psi}^{\dagger}(\boldsymbol{r}_1) + b^{*}\hat{\psi}^{\dagger}(\boldsymbol{r}_2)\right)\left(a\hat{\psi}(\boldsymbol{r}_1) + b\hat{\psi}(\boldsymbol{r}_2)\right)\right\rangle \tag{4.3}$$

where $a$ and $b$ are geometrical factors. This density is the second-quantized analogue of the single-particle probability (3.1). The coherence or Green function associated with the interference is then [4]

$$G^{(1)}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_1)\,\hat{\psi}(\boldsymbol{r}_2)\right\rangle \tag{4.4}$$

which is proportional to the single-particle density matrix [31],

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N}\left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_1)\,\hat{\psi}(\boldsymbol{r}_2)\right\rangle \tag{4.5}$$

where $N$ is the mean number of trapped atoms. This is the many-particle analogue of (3.2) and the corresponding coherence length is

$$\mathcal{L}^2 = \frac{\int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2 \left|\left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_1)\,\hat{\psi}(\boldsymbol{r}_2)\right\rangle\right|^2 (x_1 - x_2)^2}{2\int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2 \left|\left\langle \hat{\psi}^{\dagger}(\boldsymbol{r}_1)\,\hat{\psi}(\boldsymbol{r}_2)\right\rangle\right|^2}. \tag{4.6}$$

Given the coherence function (4.4) or equivalently the density matrix (4.5) we can determine the coherence length for the trapped gas. We note that our coherence length can be applied to any many-body problem including those involving fermions and those involving fewer than three dimensions. We will consider these elsewhere.

Higher-order coherence functions for a trapped Bose gas have been calculated [4–6] and measured [32, 33]. These provide information concerning density fluctuations and collisions within a condensate.

### *4.1. The density matrix*

Our aim is to calculate the coherence length for a trapped Bose gas. In order to keep our analysis as simple as possible we will describe the gas using the grand canonical ensemble in which the mean values of the energy and of the total particle number are fixed [34]. We will also restrict our discussion to the non-interacting, ideal Bose gas. Finally, we specialize to the case of an isotropic harmonic trap in which the frequencies associated with oscillation in the three Cartesian directions are equal so that the potential energy of a single trapped atom is $U(\boldsymbol{r}) = \frac{1}{2}m\omega^2\boldsymbol{r}^2$, where $m$ is the atomic mass and $\omega$ is the angular frequency of harmonic motion in the trap. The energy eigenvalues have the familiar form for a three-dimensional harmonic oscillator

$$E_{n_x,n_y,n_z} = (n_x + n_y + n_z)\hbar\omega. \tag{4.7}$$

Here we have set the zero of energy to coincide with the ground state of the trap. Naturally, this choice will also affect the value of the chemical potential.

The mean number of particles occupying any given trap state is given by the Bose–Einstein formula

$$N_{n_x,n_y,n_z} = \left\{\exp\left[\beta(E_{n_x,n_y,n_z} - \mu)\right] - 1\right\}^{-1} \tag{4.8}$$

so that the mean value of the total atom number is

$$N = \sum_{n_x,n_y,n_z=0}^{\infty} \left\{\exp\left[\beta(E_{n_x,n_y,n_z} - \mu)\right] - 1\right\}^{-1}. \tag{4.9}$$

For any given value of $N$, the (negative-valued) chemical potential $\mu$ must be chosen so that this equation is valid. Equation (4.9) can be simplified by expressing the summations over $n_x$, $n_y$ and $n_z$ in terms of a single summation over a dummy index $j$ (see [35, 36] and appendix A) to give

$$N = \sum_{j=1}^{\infty} \exp\left(j\beta\mu\right)\left[1 - \exp\left(-j\beta\hbar\omega\right)\right]^{-3}. \tag{4.10}$$

We proceed with our calculation of the density matrix (4.5) by expanding the atom annihilation and creation operators as a sum of operators for each of the trap states

$$\hat{\psi}(\boldsymbol{r}) = \sum_{n_x,n_y,n_z=0}^{\infty} u_{n_x,n_y,n_z}(\boldsymbol{r})\,\hat{a}_{n_x,n_y,n_z} \tag{4.11}$$

where the trap state annihilation and creation operators satisfy the usual commutation relations

$$[\hat{a}_{n_x,n_y,n_z}, \hat{a}^{\dagger}_{n'_x,n'_y,n'_z}] = \delta_{n_x,n'_x}\delta_{n_y,n'_y}\delta_{n_z,n'_z}. \tag{4.12}$$

In thermal equilibrium, the expectation values of products of annihilation and creation operators have the simple form

$$\left\langle \hat{a}^{\dagger}_{n'_x,n'_y,n'_z}\hat{a}_{n_x,n_y,n_z}\right\rangle = N_{n_x,n_y,n_z}\delta_{n_x,n'_x}\delta_{n_y,n'_y}\delta_{n_z,n'_z}. \tag{4.13}$$

Hence the density matrix is

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N}\sum_{n_x,n_y,n_z=0}^{\infty} u^{*}_{n_x,n_y,n_z}(\boldsymbol{r}_1)\,u_{n_x,n_y,n_z}(\boldsymbol{r}_2)\left\{\exp\left[\beta(E_{n_x,n_y,n_z} - \mu)\right] - 1\right\}^{-1} \tag{4.14}$$

which is equivalent to the coherence function given by Naraschewski and Glauber [6]. We can simplify this expression with the aid of the harmonic oscillator eigenfunctions to give (see appendix A)

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N}\left(\frac{m\omega}{\hbar}\right)^{3/2}\sum_{j=1}^{\infty}\exp(j\beta\mu)\{\pi[1 - \exp(-2j\beta\hbar\omega)]\}^{-3/2}$$
$$\times \exp\left\{-\frac{m\omega}{4\hbar}[|\boldsymbol{r}_1 + \boldsymbol{r}_2|^2\tanh(j\beta\hbar\omega/2) + |\boldsymbol{r}_1 - \boldsymbol{r}_2|^2\coth(j\beta\hbar\omega/2)]\right\}. \tag{4.15}$$

The density matrix has the form of a sum of Gaussians in the sum and difference of the coordinates, centred on the middle of the trap. In particular, the width of the Gaussian containing the sum $\boldsymbol{r}_1 + \boldsymbol{r}_2$ is proportional to the uncertainty of a single thermal particle as given in (3.6), whilst the width of the Gaussian containing the difference $\boldsymbol{r}_1 - \boldsymbol{r}_2$ is proportional to the corresponding coherence length (3.7).

We point out that the mean squared width of the Bose gas can be readily obtained from the density matrix (4.15),

$$\Delta x^2 = \int \mathrm{d}\boldsymbol{r}\,\rho(\boldsymbol{r}, \boldsymbol{r})\,x^2 = \frac{1}{N}\frac{\hbar}{2m\omega}\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)[1 + \exp(-j\beta\hbar\omega)]}{[1 - \exp(-j\beta\hbar\omega)]^4}. \tag{4.16}$$

In the zero-temperature limit, $\beta \to \infty$ and the width of the ground state is given by

$$\Delta x_0^2 = \frac{\hbar}{2m\omega}. \tag{4.17}$$

For large temperatures the number of atoms occupying any given trap state is small and the density matrix (4.15) approaches the simpler form (A.14). In this limit the width of the thermal cloud becomes

$$\Delta x_{\mathrm{thermal}}^2 = \frac{\hbar}{2m\omega}\coth(\beta\hbar\omega/2) \tag{4.18}$$

which is the same as the single-particle expression (3.6) and grows approximately linearly with temperature.

### 4.2. Coherence length

The coherence length can be obtained by substituting the density matrix (4.15) into (4.6). We present, in appendix A, the evaluation of the required integrals and summations and show that the coherence length can be written in the form

$$\mathcal{L}^2 = \frac{\hbar}{m\omega}\left\{\frac{1}{2} + \sum_{n=2}^{\infty} \mathrm{e}^{n\beta\mu}\left[1-\mathrm{e}^{-n\beta\hbar\omega}\right]^{-4}\left[(n-1)\mathrm{e}^{-n\beta\hbar\omega} - (1-\mathrm{e}^{-(n-1)\beta\hbar\omega})(\mathrm{e}^{\beta\hbar\omega}-1)^{-1}\right]\right.$$
$$\left.\times\left[\sum_{n=2}^{\infty}(n-1)\mathrm{e}^{n\beta\mu}[1-\mathrm{e}^{-n\beta\hbar\omega}]^{-3}\right]^{-1}\right\}. \tag{4.19}$$

In the zero-temperature limit, $\beta \to \infty$ and the ratio of summations tends to zero so we are left with

$$\mathcal{L}_0^2 = \frac{\hbar}{2m\omega} \tag{4.20}$$

which is the value associated with the width of the trap ground state. This equivalence between the condensate coherence length and the size of the condensate has been demonstrated in [7]. More generally, the coherence length is an explicit function of the inverse temperature $\beta$ and of the chemical potential $\mu$, but depends on the total number of atoms $N$ only implicitly through $\mu$. We are interested in the temperature dependence of the coherence length for a given number of atoms and therefore substitute into (4.19) the appropriate value of $\mu$ determined from (4.10).

Of particular interest is the behaviour of the coherence length in the vicinity of the critical temperature

$$T_{\mathrm{C}} = \frac{\hbar\omega}{k_{\mathrm{B}}}\left(\frac{N}{\zeta(3)}\right)^{1/3} \approx 0.941\frac{\hbar\omega}{k_{\mathrm{B}}}N^{1/3} \tag{4.21}$$

below which condensation occurs [37–39]. Here, the Riemann zeta-function is defined as $\zeta(t) = \sum_{n=1}^{\infty} n^{-t}$. The critical temperature can be interpreted as the temperature at which the de Broglie waves for the individual atoms overlap [40]. It can also be understood as a consequence of the uncertainty principle [41].

In figure 1(*a*) we plot the ratio $\mathcal{L}/\mathcal{L}_0$ as a function of temperature for a mean total number of atoms $N = 1$, 10, $10^2$, $10^3$ and $10^4$, indicated by the broken curves. The temperature has been scaled to reflect the formation of a condensate below the critical temperature. For comparison, the fraction of atoms in the (condensed) ground state is shown broken in figure 1(*b*). In the limit $N \to \infty$, the asymptotic form of the coherence length is a Heaviside function in temperature

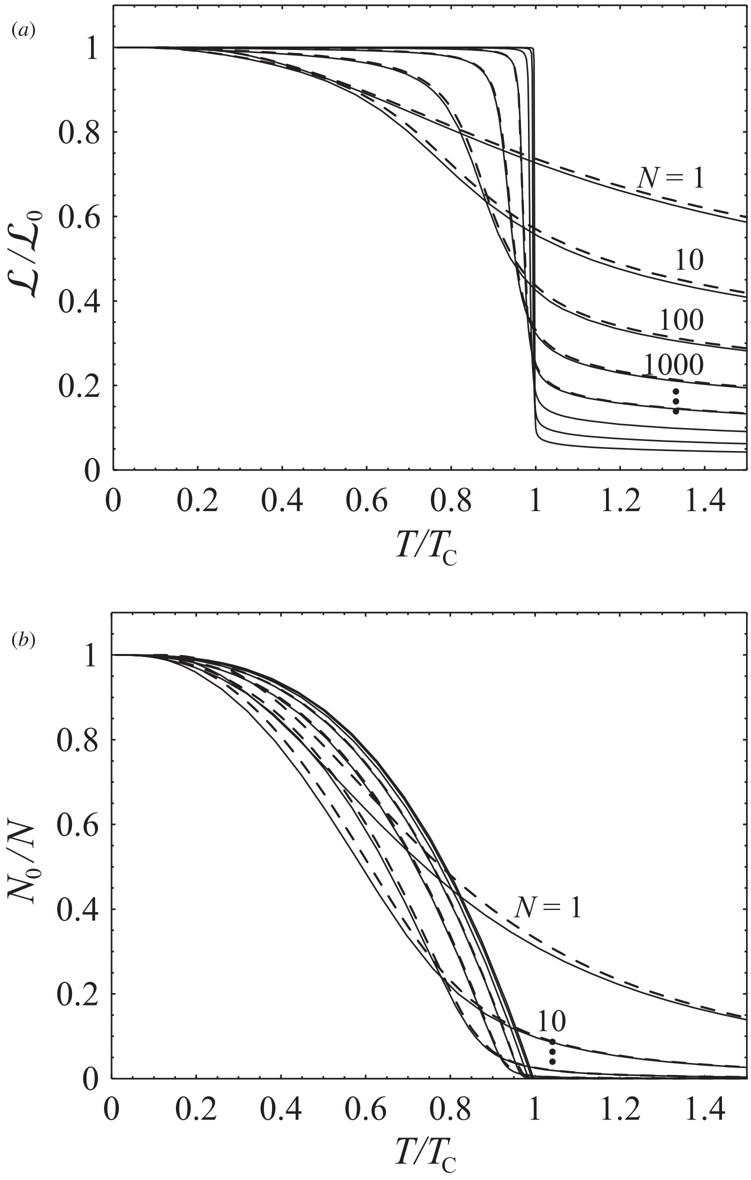


**Figure 1.** (*a*) The normalized coherence length and (*b*) ground-state occupation as a function of temperature in units of the critical temperature for atom numbers between 1 and $10^7$ in factors of 10. Broken curves indicate the exact calculations using (4.19) and (4.10), which were only performed for atom numbers up to $10^4$. Full curves represent the approximate forms (A.10) and (A.9).

with a transition from $\mathcal{L} = \mathcal{L}_0$ to $\mathcal{L} = 0$ at the critical temperature, whilst below $T = T_C$ the ground state occupation can be described by the equation [38]

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^3. \tag{4.22}$$

Remarkably, the coherence length shows a significant increase as soon as condensation starts to occur, even when only a small fraction of atoms is in the ground state. This is a consequence of

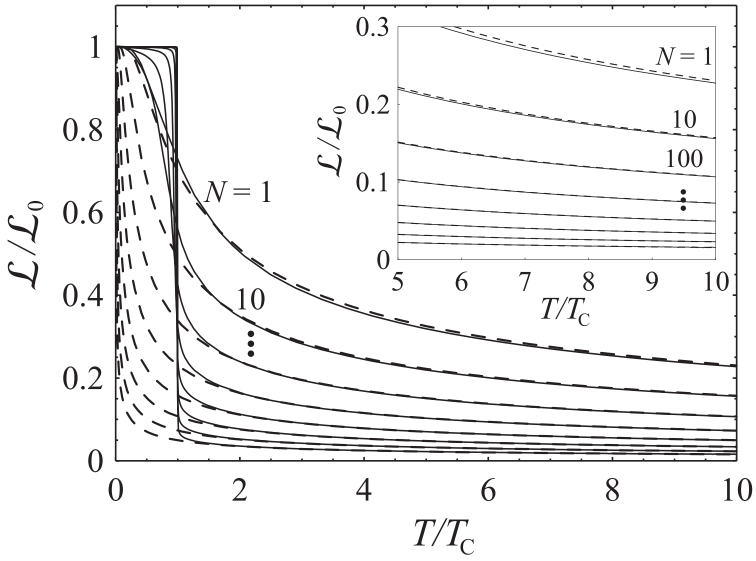


**Figure 2.** The normalized coherence length (A.10) for higher temperatures. For temperatures above the critical value the coherence length approaches the value for a single thermal particle (broken curves). This can be seen more clearly in the magnified plot region (inset).

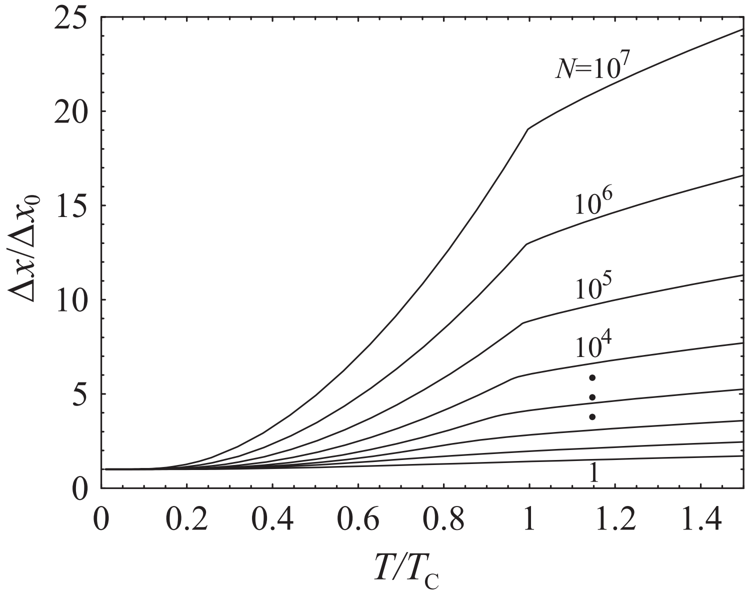


**Figure 3.** The normalized RMS width of the Bose gas as a function of temperature for atom numbers between 1 and $10^7$ in factors of 10. We have used the approximate form (A.12).

the change in the first-order coherence function associated with condensation, which has been noted previously [4, 6, 9]. The larger the number of atoms, the more abruptly the coherence length rises to its maximal (zero-temperature) value as the temperature decreases below $T_C$.

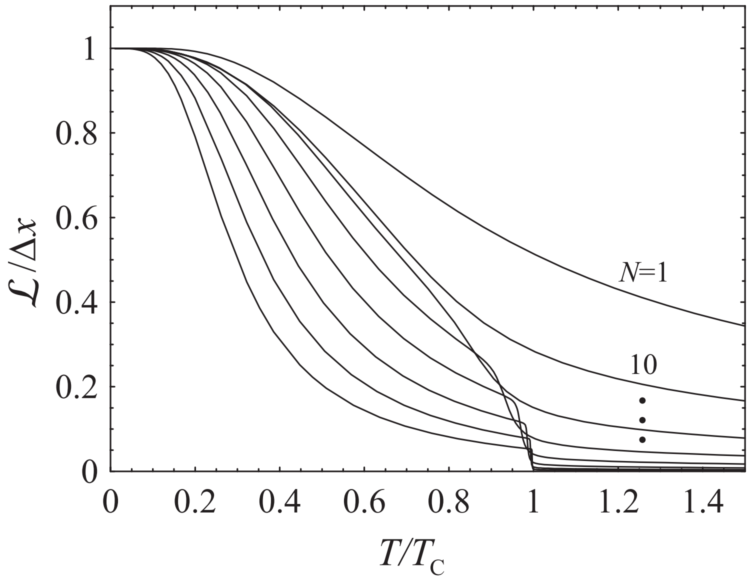


**Figure 4.** The ratio of coherence length to the Bose gas width as a function of temperature for atom numbers between 1 and $10^7$ in factors of 10.

In order to study the coherence length for larger numbers of atoms, we employ an approximate form (A.10) derived in appendix A. This approximate form is plotted as full curves in figure 1(*a*), for mean atom numbers increasing by factors of 10 from 1 to $10^7$. The corresponding approximation for the fraction of atoms in the ground state (A.9) is shown in figure 1(*b*). For large numbers of atoms the coherence length remains near to its zero-temperature value up to temperatures just below $T_C$, even though only a small fraction of atoms are condensed. Above $T_C$ the coherence length is greatly reduced.

At high temperatures the gas is no longer degenerate and we can neglect the unity term in the denominator of the Bose formula (4.8). We show in appendix A, that in this limit the coherence length reduces to its single-particle form (3.6). In figure 2 we compare the coherence length (full curves) with this high-temperature approximation (broken curves). It is clear that the high-temperature approximation, which ignores degeneracy, provides a good approximation for temperatures above $T_C$.

It is important to remember that, although at temperatures just below $T_C$ the coherence length quickly rises to $\mathcal{L}_0$, the total width of the condensate greatly exceeds its ground state width of $\Delta x_0 = \mathcal{L}_0$, as shown in figure 3. The trapped gas' coherence length as a fraction of the cloud width (figure 4) gives a clearer perspective of the relative length scales as the transition temperature is traversed. Even in this 'normalized' picture the coherence length increases abruptly at $T_C$ for large atom numbers.

## 5. Conclusion

The coherence length is a critical measure for the observation of interference. In this paper we have derived its form for a trapped ideal Bose gas.

We have demonstrated that the onset of condensation coincides with a dramatic increase in the coherence length, especially for a large number of atoms. This increase is remarkable as the coherence length reaches its maximum value at temperatures for which only a small proportion of the trapped atoms are in the ground state. The increase in coherence length provides a simple explanation for the interference between spatially separated regions of a condensate just below the critical temperature [3].

Interference and coherence are often associated with the existence of a pure state for the trapped gas. This description is indeed valid for temperatures close to zero, but does not apply in the vicinity of the critical temperature. The least-squares fit of appendix B shows that the nearest pure-state approximation to the density matrix (4.15) is the trap ground state (B.11) with a mean occupation number $N_0$. This means that a pure-state description is not applicable just below the critical temperature when $N_0 \ll N$. Thus, although a pure state will give rise to coherence, it is, of course, perfectly possible for coherent phenomena to be observed for mixed states.

It should be emphasized that the spatial extent of the trapped gas, as measured by (4.16), is larger than the coherence length. Only in the limit of zero temperature do they tend to the same value. This means that there is a jump in partial coherence at the critical temperature. Coherent phenomena will be more apparent below the critical temperature, but are not expected to extend across the whole of the trapped gas until much lower temperatures are reached.

The coherence length may also be calculated for other many-body problems including anisotropic gases, Bose gases in one and two dimensions and trapped Fermi gases. We intend to investigate such cases in the future.

**Acknowledgments**

We thank Guillaume Huyet for numerous enlightening discussions on coherence length, Claire Gilson for proposing the least-squares fit to a wavefunction and for providing the proof presented in appendix B and John Jeffers for helpful discussions. This work was supported by the United Kingdom Engineering and Physical Sciences Research Council and by the TMR programme of the Commission of the European Union through the Quantum Structures Network. ASA thanks the Royal Society of Edinburgh and the Scottish Office Education and Industry Department for the award of a Research Fellowship.

## Appendix A. Mathematical details

We present in this appendix some of the more involved mathematical derivations of the formulae presented in the text.

The Bose–Einstein formula for the mean total atom number (4.9) can be replaced by a simpler expression involving only a single sum (4.10). This can be achieved by writing (4.9) in the form [35, 36]

$$
\begin{aligned}
N &= \sum_{n_x,n_y,n_z=0}^{\infty} \sum_{j=1}^{\infty} \exp\left[-j\beta(E_{n_x,n_y,n_z} - \mu)\right] \\
&= \sum_{j=1}^{\infty} \exp(j\beta\mu) \left[\sum_{n=0}^{\infty} \exp(-jn\beta\hbar\omega)\right]^3 \\
&= \sum_{j=1}^{\infty} \exp(j\beta\mu) \left[1 - \exp(-j\beta\hbar\omega)\right]^{-3}.
\end{aligned}
\tag{A.1}
$$

We can obtain the density matrix in the form (4.15) by first introducing an additional sum as in (A.1) and by giving explicit expressions for the trap state eigenfunctions. This leads us to rewrite (4.14) in the form

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N}\left(\frac{m\omega}{\hbar}\right)^{3/2} \sum_{j=1}^{\infty} \mathrm{e}^{j\beta\mu} \left[\sum_{n=0}^{\infty} \mathrm{e}^{-jn\beta\hbar\omega} \frac{\exp[-\frac{1}{2}(X_1^2 + X_2^2)]}{2^n n! \pi^{1/2}} H_n(X_1) H_n(X_2)\right] \times [X \to Y][X \to Z] \qquad \text{(A.2)}$$

where we have introduced dimensionless coordinates

$$X = \left(\frac{m\omega}{\hbar}\right)^{1/2} x \qquad \text{(A.3)}$$

and the final terms in square brackets indicate replacing $X$ by $Y$ and by $Z$. We can evaluate the sum over $n$ in (A.2) by using the integral representation of the Hermite polynomials and their generating function [42] to give

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{e}^{-jn\beta\hbar\omega} \frac{\exp\left[-\frac{1}{2}\left(X_1^2 + X_2^2\right)\right]}{2^n n! \pi^{1/2}} H_n(X_1) H_n(X_2) \\ &\quad = \pi^{-1} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-t^2} \mathrm{e}^{-(X_1^2+X_2^2)/2} \sum_{n=0}^{\infty} (n!)^{-1} \left(X_1 + \mathrm{i}t\right)^n \mathrm{e}^{-jn\beta\hbar\omega} H_n(X_2) \\ &\quad = \pi^{-1} \mathrm{e}^{-(X_1^2+X_2^2)/2} \int_{-\infty}^{\infty} \mathrm{d}t \exp\left[-t^2 - \mathrm{e}^{-2j\beta\hbar\omega}(X_1 + \mathrm{i}t)^2 + 2\mathrm{e}^{-j\beta\hbar\omega} X_2 (X_1 + \mathrm{i}t)\right]. \end{aligned} \qquad \text{(A.4)}$$

Evaluating the Gaussian integral then gives

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{e}^{-jn\beta\hbar\omega} \frac{\exp\left[-\frac{1}{2}\left(X_1^2 + X_2^2\right)\right]}{2^n n! \pi^{1/2}} H_n(X_1) H_n(X_2) \\ &\quad = \pi^{-1/2} \left(1 - \mathrm{e}^{-2j\beta\hbar\omega}\right)^{-1/2} \exp\left\{-\tfrac{1}{4}\left[\, (X_1 + X_2)^2 \tanh\left(j\beta\hbar\omega/2\right) \right.\right. \\ &\qquad \left.\left. + (X_1 - X_2)^2 \cosh\left(j\beta\hbar\omega/2\right)\,\right]\right\}. \end{aligned} \qquad \text{(A.5)}$$

Applying the same procedure to the $Y$ and $Z$ summations and reinstating the original coordinates gives the density matrix in the form (4.15).

The coherence length (4.19) may be obtained by inserting the density matrix (4.15) into our formula for the coherence length (4.6) and evaluating the integrals. For the denominator we find

$$\begin{aligned} 2 \int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2\, |\rho(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 &= \frac{1}{4N^2}\left(\frac{4\hbar}{m\omega}\right)^3 \sum_{j,l=1}^{\infty} \frac{\mathrm{e}^{(j+l)\beta\mu}}{(1 - \mathrm{e}^{-2j\beta\hbar\omega})^{3/2}(1 - \mathrm{e}^{-2l\beta\hbar\omega})^{3/2}} \\ &\quad \times [\tanh(j\beta\hbar\omega/2) + \tanh(l\beta\hbar\omega/2)]^{3/2} [\cosh(j\beta\hbar\omega/2) + \cosh(l\beta\hbar\omega/2)]^{3/2} \\ &= \frac{16}{N^2}\left(\frac{\hbar}{m\omega}\right)^3 \sum_{j,l=1}^{\infty} \mathrm{e}^{(j+l)\beta\mu} \left[1 - \mathrm{e}^{-(j+l)\beta\hbar\omega}\right]^{-3} \\ &= \frac{16}{N^2}\left(\frac{\hbar}{m\omega}\right)^3 \sum_{n=2}^{\infty} (n-1)\mathrm{e}^{n\beta\mu} \left[1 - \mathrm{e}^{-n\beta\hbar\omega}\right]^{-3}. \end{aligned} \qquad \text{(A.6)}$$

For the numerator we find

$$\int \mathrm{d}\boldsymbol{r}_1 \int \mathrm{d}\boldsymbol{r}_2\, |\rho(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 (x_1 - x_2)^2 = \frac{8}{N^2}\left(\frac{\hbar}{m\omega}\right)^3 \sum_{j,l=1}^{\infty} \mathrm{e}^{(j+l)\beta\mu}\left[1 - \mathrm{e}^{-(j+l)\beta\hbar\omega}\right]^{-3}$$

$$\times \frac{2}{\coth(j\beta\hbar\omega/2) + \coth(l\beta\hbar\omega/2)}$$

$$= \frac{8}{N^2}\left(\frac{\hbar}{m\omega}\right)^3 \sum_{j,l=1}^{\infty} \mathrm{e}^{(j+l)\beta\mu}\left[1 - \mathrm{e}^{-(j+l)\beta\hbar\omega}\right]^{-4}\left(1 - \mathrm{e}^{-j\beta\hbar\omega}\right)\left(1 - \mathrm{e}^{-l\beta\hbar\omega}\right)$$

$$= \frac{8}{N^2}\left(\frac{\hbar}{m\omega}\right)^3 \sum_{n=2}^{\infty} \mathrm{e}^{n\beta\mu}\left[1 - \mathrm{e}^{-n\beta\hbar\omega}\right]^{-4}$$

$$\times \left[(n-1)\left(1 + \mathrm{e}^{-n\beta\hbar\omega}\right) - 2\left(1 - \mathrm{e}^{-(n-1)\beta\hbar\omega}\right)\left(\mathrm{e}^{\beta\hbar\omega} - 1\right)^{-1}\right]. \qquad \text{(A.7)}$$

The ratio of (A.7) and (A.6) is the square of the coherence length given in (4.19).

We can obtain a very good approximation to the atom number (4.10) and the coherence length (4.19) by making the replacement

$$\frac{1}{1 - \mathrm{e}^{-x}} \approx 1 + \frac{\mathrm{e}^{-x/2}}{x} \qquad \text{(A.8)}$$

which is accurate to better than 2.5% for all positive $x$ [35, 43]. The resulting approximate forms for the atom number and coherence length are

$$N = \sum_{j=0}^{3} \binom{3}{j} (\beta\hbar\omega)^{-j}\, g_j\left(\mathrm{e}^{\beta(\mu - j\hbar\omega/2)}\right) \qquad \text{(A.9)}$$

$$\frac{\mathcal{L}^2}{\mathcal{L}_0^2} = 1 + 2\sum_{j=0}^{4} \binom{4}{j} (\beta\hbar\omega)^{-j}$$

$$\times\left\{g_{j-1}\left(\mathrm{e}^{\beta[\mu-(1+j/2)\hbar\omega]}\right) + \left[g_j(\mathrm{e}^{\beta[\mu-(1+j/2)\hbar\omega]}) - g_j(\mathrm{e}^{\beta[\mu - j\hbar\omega/2]})\right]/(\mathrm{e}^{\beta\hbar\omega} - 1)\right\}$$

$$\times\left\{\sum_{j=0}^{3} \binom{3}{j} (\beta\hbar\omega)^{-j}\left[g_{j-1}(\mathrm{e}^{\beta[\mu - j\hbar\omega/2]}) - g_j(\mathrm{e}^{\beta[\mu - j\hbar\omega/2]})\right]\right\}^{-1} \qquad \text{(A.10)}$$

where we have used the Bose functions [34]

$$g_k(z) = \sum_{n=1}^{\infty} z^n/n^k. \qquad \text{(A.11)}$$

Similarly, the width of the Bose gas (4.16) can be approximated by

$$\Delta x^2 = \frac{1}{N}\frac{\hbar}{2m\omega}\sum_{j=0}^{4}\binom{4}{j}(\beta\hbar\omega)^{-j}\left[g_j(\mathrm{e}^{\beta(\mu - j\hbar\omega/2)}) + g_j(\mathrm{e}^{\beta[\mu-(1+j/2)\hbar\omega]})\right]. \qquad \text{(A.12)}$$

To obtain the high-temperature limit of the coherence length we note that at high temperatures there is only a small probability of finding more than one atom in any given trap state. This means that we can neglect the 1 in the Bose formula (4.8) [41]. In this approximation the density matrix (4.14) becomes

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{e^{\beta\mu}}{N} \sum_{n_x,n_y,n_z=0}^{\infty} u^*_{n_x,n_y,n_z}(\boldsymbol{r}_1)\, u_{n_x,n_y,n_z}(\boldsymbol{r}_2) \exp[-\beta\hbar\omega(n_x + n_y + n_z)]$$

$$= \frac{e^{\beta\mu}}{N} \left(\frac{m\omega}{\hbar}\right)^{3/2} \left[\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} \frac{\exp[-\frac{1}{2}(X_1^2 + X_2^2)]}{2^n n! \pi^{1/2}} H_n(X_1) H(X_2)\right]$$

$$\times [X \to Y][X \to Z] \tag{A.13}$$

where we have again used the dimensionless coordinates (A.3). The remaining summations can be evaluated using (A.4) and (A.5) to give

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{e^{\beta\mu}}{N} \left(\frac{m\omega}{\hbar}\right)^{3/2} \left[\pi\left(1 - e^{-2\beta\hbar\omega}\right)\right]^{-3/2}$$

$$\times \exp\left\{-\frac{m\omega}{4\hbar}\left[|\boldsymbol{r}_1 + \boldsymbol{r}_2|^2 \tanh(\beta\hbar\omega/2) + |\boldsymbol{r}_1 - \boldsymbol{r}_2|^2 \coth(\beta\hbar\omega/2)\right]\right\}. \tag{A.14}$$

Inserting this into (4.6) gives

$$\mathcal{L}^2 = \frac{\hbar}{2m\omega} \tanh(\beta\hbar\omega/2) \tag{A.15}$$

which is of the same form as found for a single trapped particle (3.6).

## Appendix B. Least-squares fit to a wavefunction

The existence of off-diagonal long-range order has long been recognized as an important, even defining feature of Bose–Einstein condensation [9–11]. This is associated with the approximate factorization of the density matrix as

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N}\left\langle \hat{\psi}^\dagger(\boldsymbol{r}_1)\, \hat{\psi}(\boldsymbol{r}_2)\right\rangle = \frac{1}{N} \psi^*(\boldsymbol{r}_1)\, \psi(\boldsymbol{r}_2) + \text{small terms}. \tag{B.1}$$

The function $\psi(\boldsymbol{r})$ is associated with the condensed atoms in the ground state. Here we show by means of a least-squares fit that the best approximate density matrix of the form

$$\rho_{\text{approx}}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N} \psi^*(\boldsymbol{r}_1)\, \psi(\boldsymbol{r}_2) \tag{B.2}$$

is that for the ground state of the trap.

The least-squares fit between (B.1) and (B.2) corresponds to finding the minimum value of the quantity

$$M = \int d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 \left[\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) - \rho_{\text{approx}}(\boldsymbol{r}_1, \boldsymbol{r}_2)\right]^2. \tag{B.3}$$

In order to evaluate $M$ we expand the density matrix and the approximate wavefunction in terms of the trap states:

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{N} \sum_{\{n\}=0}^{\infty} u^*_{\{n\}}(\boldsymbol{r}_1)\, u_{\{n\}}(\boldsymbol{r}_2)\, N_{\{n\}} \tag{B.4}$$

$$\psi(\boldsymbol{r}) = \sum_{\{n\}=0}^{\infty} \psi_{\{n\}} u_{\{n\}}(\boldsymbol{r}). \tag{B.5}$$

Here we denote by $\{n\}$ the set of three quantum numbers $n_x$, $n_y$ and $n_z$. Inserting (B.4) and (B.5) into (B.3) and using the orthonormality of the trap states,

$$\int \mathrm{d}\boldsymbol{r}\, u^*_{\{n\}}(\boldsymbol{r})\, u_{\{n'\}}(\boldsymbol{r}) = \delta_{\{n\},\{n'\}} = \delta_{n_x,n'_x}\delta_{n_y,n'_y}\delta_{n_z,n'_z} \tag{B.6}$$

gives

$$M = \sum_{\{n\},\{n'\}} \left(N_{\{n\}}\delta_{\{n\},\{n'\}} - |\psi_{\{n\}}||\psi_{\{n'\}}|\right)^2. \tag{B.7}$$

Reordering the terms in the summations leads to

$$M = \sum_{\{n\}\neq\{0\}} N^2_{\{n\}} + \left[N_{\{0\}} - \sum_{\{n\}} |\psi_{\{n\}}|^2\right]^2 + 2\sum_{\{n\}} (N_{\{0\}} - N_{\{n\}})|\psi_{\{n\}}|^2 \tag{B.8}$$

where $N_{\{0\}}$ represents the ground state $N_0$. Each of the three terms in (B.8) is positive by virtue of the fact that the mean population (4.8) is the greatest in the ground state so that $N_{\{n\}} \geqslant N_0$. The first term in $M$ is fixed by the Bose statistics of the density matrix. The minimum value of each of the two remaining terms is zero so that

$$M_{\min} = \sum_{\{n\}\neq\{0\}} N^2_{\{n\}} = \sum_{\{n\}} N^2_{\{n\}} - N_0^2. \tag{B.9}$$

This minimum value is obtained by setting

$$|\psi_{\{n\}}|^2 = N_0\delta_{n_x,0}\delta_{n_y,0}\delta_{n_z,0} \tag{B.10}$$

so that $\psi(\boldsymbol{r})$ is the trap ground state

$$\psi(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\phi}\sqrt{N_0}\, u_{0,0,0}(\boldsymbol{r}) \tag{B.11}$$

where $\phi$ is an arbitrary, constant phase. Note that this wavefunction is not normalized to the total number of trapped atoms but to the number of atoms trapped in the ground state:

$$\int \mathrm{d}\boldsymbol{r}\, |\psi(\boldsymbol{r})|^2 = N_0. \tag{B.12}$$

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