

Energy-momentum balance in quantum dielectrics

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We calculate the energy-momentum balance in quantum dielectrics such as Bose-Einstein condensates. In agreement with the experiment [G. K. Campbell *et al.*, Phys. Rev. Lett. **94**, 170403 (2005)] variations of the Minkowski momentum are imprinted onto the phase, whereas the Abraham tensor drives the flow of the dielectric. Our analysis indicates that the Abraham-Minkowski controversy has its root in the Röntgen interaction of the electromagnetic field in dielectric media.

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I. INTRODUCTION

It is surprising [1] that the momentum of light in media has been subject to considerable debate [1–4] for almost a century. Moreover, although the main contenders, the Minkowski momentum [2] $\mathbf{D} \times \mathbf{B}$ and the Abraham momentum [3] $\mathbf{E} \times \mathbf{H}/c^2$, differ by a substantial factor, the refractive index squared, precise experimental tests of this thorny issue have been scarce [5,6]. Only recently, the photon recoil momentum was directly measured [6] in a medium made of a Bose-Einstein condensate [7]. From a theoretical point of view, a condensate is a very simple system, a nearly ideal quantum gas or irrotational fluid [7] that allows studies of fundamental effects without many complications from material details. From a practical perspective, the momentum transfer of light in condensates is important in high-precision atom interferometry [6].

Here we deduce the energy-momentum balance in such quantum dielectrics from geometric principles that do not depend much on microscopic details. We find that, in the nonrelativistic limit, for a condensate with number density ϱ , phase S , electric permittivity ε and magnetic permeability μ the total momentum density is

$$\mathbf{g} = \varrho \hbar \nabla S + \mathbf{D} \times \mathbf{B}. \quad (1)$$

This result shows that the Minkowski momentum is imprinted into the phase of the quantum dielectric, in agreement with the experiment [6]. However, we can also express the total momentum density in the Abraham form

$$\mathbf{g} = \varrho m \mathbf{u} + \frac{\mathbf{E} \times \mathbf{H}}{c^2}, \quad (2)$$

where m is the atomic mass, defining the quantity

$$\mathbf{u} = \frac{1}{m} \left[\hbar \nabla S + \left(\varepsilon - \frac{1}{\mu} \right) \frac{\mathbf{E} \times \mathbf{B}}{\varrho} \right] \quad (3)$$

and using the constitutive equations of the electromagnetic field in a medium at rest,

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}, \quad \mathbf{H} = \frac{\varepsilon_0 c^2}{\mu} \mathbf{B}. \quad (4)$$

The important point of this simple exercise in reexpressing \mathbf{g} is the physical meaning of \mathbf{u} : For dielectric media, \mathbf{u} de-

scribes the flow velocity. The mechanical momentum $m\mathbf{u}$ differs from the canonical momentum $\hbar \nabla S$ by the Röntgen term $(\varepsilon - 1/\mu)\mathbf{E} \times \mathbf{B}/\varrho$ that, for neutral atoms, plays the role of the vector potential for charged particles [8].

The Röntgen interaction [8] stems from the behavior of the moving atomic dipoles that constitute the medium: When set in motion induced electric dipoles perceive mixtures of electric and magnetic fields, according to the Lorentz transformations of the fields [9], and the same applies to magnetic dipoles. The Röntgen interaction [8] is most easily deduced from the Lagrangian of a classical induced dipole

$$L = \frac{m}{2} u^2 + \frac{\alpha_E}{2} E'^2 + \frac{\alpha_B}{2} c^2 B'^2, \quad (5)$$

where α_E and α_B denote the electric and magnetic polarizabilities. The electromagnetic field interacts with the moving dipole in the locally comoving frame, indicated by primes, where the dipole is at rest. In lowest order, we obtain from the Lorentz transformations [9], $\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$, $\mathbf{B}' = \mathbf{B} - \mathbf{u} \times \mathbf{E}/c^2$ the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = m\mathbf{u} - (\alpha_E + \alpha_B)(\mathbf{E} \times \mathbf{B}). \quad (6)$$

For a medium with dipole density ϱ , the electric and magnetic polarizabilities give rise to the electric permittivity and magnetic permeability

$$\varepsilon = 1 + \frac{\alpha_E}{\varepsilon_0} \varrho, \quad \mu = 1 - \frac{\alpha_B}{\varepsilon_0} \varrho. \quad (7)$$

For a quantum dielectric, the canonical momentum corresponds to the phase gradient, which gives Eq. (3).

Formulas (1)–(3) illustrate the ambivalent nature of the electromagnetic momentum in media: Variations of the Minkowski momentum are imprinted onto the phase, whereas the Abraham momentum drives the flow of the quantum dielectric. Flow and phase gradient differ by the Röntgen term. Consequently, the Röntgen interaction [8] appears to be at the heart of the Abraham-Minkowski controversy [4].

In this paper, we deduce and generalize the relationships (1)–(3) in a relativistically covariant form. For our analysis we borrow ideas from general relativity [10] that have their historic precursors in Fermat's Principle in geometric optics

[11]: Light rays take the shortest optical paths in a medium, regardless how curved are their trajectories. In terms of general relativity [10], the rays follow geodesic lines with respect to a metric that is proportional to the refractive index. In general, the electromagnetic field perceives isotropic, non-dispersive and possibly moving media as effective space-time geometries [12,13]. On the other hand, in atom optics the roles of light and matter are often reversed: Light acts as a medium for matter waves. It is therefore natural to postulate [13] that quantum dielectrics perceive the electromagnetic field as an effective space-time geometry as well. This approach turned out to be completely consistent with Maxwell's equations [13], which justifies the idea. However, our previous calculation of the energy-momentum tensor was indirect, flawed [14], and gave the wrong energy-momentum tensor of the dielectric matter. Here we calculate the energy-momentum balance directly and show how the Minkowski and Abraham forms are related to each other, considering the dynamics of the medium. In Sec. II we summarize the essence of the geometric approach to quantum dielectrics. Section III deduces the Minkowski form of the energy-momentum balance, both proving and generalizing the result (1) in the nonrelativistic limit. Section IV casts the total energy-momentum tensor in Abraham form, expressing the general connection between the two forms.

II. GEOMETRIC APPROACH

For simplicity, we entirely focus on the dielectric interaction of the atomic condensate with the electromagnetic field. The collisional contact interaction of the atoms and any external potentials can be easily included as additional terms in the energy-momentum balance and hence they are not considered here. We use the relativistic notation of Ref. [13] with Greek indices referring to the space-time coordinates x^α , but we restrict ourselves to Cartesian coordinates in flat Minkowski space-time with metric tensor $g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$. Partial derivatives with respect to the coordinates x^α are denoted by ∂_α . Throughout this paper we employ Einstein's summation convention over repeated indices. The electromagnetic field is described by the antisymmetric field-strength tensor $F_{\alpha\beta}$ in SI units [13].

We require that the quantum dielectric perceives the electromagnetic field as an effective space-time geometry with the atomic metric tensor $g_{\alpha\beta}^A$ that is relativistically covariant. In locally comoving Galilean coordinates $g_{\alpha\beta}^A$ contains the dipole potentials of the atoms, $-(\alpha_E/2)E^2$ and $-(\alpha_B/2)c^2B^2$. Consequently, $g_{\alpha\beta}^A$ must be quadratic in the field strengths. This, combined with general covariance, leads to the only option [13]

$$g_{\alpha\beta}^A = (1 - a\mathcal{L}_F)g_{\alpha\beta} - bT_{\alpha\beta}^F, \quad (8)$$

where \mathcal{L}_F denotes the free-field Lagrangian density [10] and $T_{\alpha\beta}^F$ the free-energy-momentum tensor [10],

$$\mathcal{L}_F = -\frac{\epsilon_0}{4}F_{\alpha\beta}F^{\alpha\beta},$$

$$T_{\alpha\beta}^F = \epsilon_0 F_{\alpha\alpha'} g^{\alpha'\beta'} F_{\beta'\beta} - \mathcal{L}_F g_{\alpha\beta}. \quad (9)$$

We consider the condensate in the hydrodynamic limit [7] subject to the standard Lagrangian density

$$\mathcal{L}_A = \sqrt{-g_A} |\psi|^2 \left(\frac{\hbar^2}{2m} g_A^{\alpha\beta} (\partial_\alpha S)(\partial_\beta S) - \frac{mc^2}{2} \right). \quad (10)$$

Here g_A is the determinant and $g_A^{\alpha\beta}$ the inverse of the metric tensor $g_{\alpha\beta}^A$. The coefficients a and b in $g_{\alpha\beta}^A$ are obtained from considering the nonrelativistic limit of the atomic Lagrangian density (10). They turn out to be related to the electric and magnetic polarizabilities α_E and α_B as [13]

$$a = \frac{\alpha_E - \alpha_B}{\epsilon_0 mc^2}, \quad b = \frac{\alpha_E + \alpha_B}{\epsilon_0 mc^2}. \quad (11)$$

The Euler-Lagrange equations give the Hamilton-Jacobi equation

$$g_A^{\alpha\beta} (\partial_\alpha S)(\partial_\beta S) = \frac{m^2 c^2}{\hbar^2} \quad (12)$$

and the equation of continuity

$$\partial_\alpha \sqrt{-g_A} |\psi|^2 g_A^{\alpha\beta} \partial_\beta S = 0. \quad (13)$$

Consider the four vectors

$$w_\alpha = \frac{\hbar}{mc} \partial_\alpha S, \quad w^{(\alpha)} = g_A^{\alpha\beta} w_\beta. \quad (14)$$

If $g_A^{\alpha\beta}$ were the true space-time metric in which the medium moves $w^{(\alpha)}$ would be the four velocity of the dielectric, because it enters as a four velocity in the equation of continuity (13) and is normalized to unity with respect to the metric (8). We obtain from the Hamilton-Jacobi Eq. (12) the geodesic equation

$$w^{(\alpha)} \partial_\alpha w_\nu = \frac{1}{2} w^{(\alpha)} w^{(\beta)} \partial_\nu g_{\alpha\beta}^A \quad (15)$$

that plays the role of the Euler equation of the dielectric fluid [15]. However, the true four-velocity u^α differs from $w^{(\alpha)}$ by the norm of $w^{(\alpha)}$ with respect to the metric $g_{\alpha\beta}$.

$$u^\alpha = \frac{w^{(\alpha)}}{w}, \quad w = \sqrt{g_{\alpha\beta} w^{(\alpha)} w^{(\beta)}}. \quad (16)$$

In a realistic dielectric, the norm w of the quasifour-velocity $w^{(\alpha)}$ is nearly unity, up to corrections in the order of the electromagnetic field energy divided by mc^2 [13]. Note that such corrections matter in expressions that contain the total energy including the rest energy. The equations of continuity imply that the atom-number density ϱ is not $|\psi|^2$, but

$$\varrho = \sqrt{-g_A} w |\psi|^2. \quad (17)$$

In the nonrelativistic limit, Eq. (16) reduces to the relationship (3) between the three-dimensional velocity \mathbf{u} and the gradient of the phase S including the Röntgen term. In other words, the difference between co- and contravariant velocity vectors in the effective geometry (8) accounts for the Röntgen interaction [8].

Finally, we note that our geometric approach is justified [13], because the total Lagrangian density $\mathcal{L}_A + \mathcal{L}_F$ that generates the equations of motion (12) and (13) also generates Maxwell's equations

$$\partial_\alpha \mathbf{H}^{\alpha\beta} = 0. \quad (18)$$

Here $H^{\alpha\beta}$ contains the electromagnetic \mathbf{D} and \mathbf{H} fields in SI units [13] with the constitutive equations

$$H^{\alpha\beta} = \frac{\epsilon_0}{\mu} g_F^{\alpha\alpha'} g_F^{\beta\beta'} F_{\alpha'\beta'} \quad (19)$$

expressed in terms of Gordon's metric [12,13]

$$g_F^{\alpha\alpha'} = g^{\alpha\alpha'} + (\epsilon\mu - 1)u^\alpha u^{\beta'} \quad (20)$$

with the local four-velocity (16) and the electric permittivity ϵ and magnetic permeability μ

$$\begin{aligned} \epsilon &= 1 + \frac{a+b}{2} mc^2 w \varrho = 1 + \frac{\alpha_E}{\epsilon_0} w \varrho, \\ \frac{1}{\mu} &= 1 + \frac{a-b}{2} mc^2 w \varrho = 1 - \frac{\alpha_B}{\epsilon_0} w \varrho, \end{aligned} \quad (21)$$

that are consistent with the polarizabilities (7) in the realistic case where w approaches unity. Equations (19) and (20) correspond to the constitutive Eqs. (4) in locally comoving frames. Light and matter perceive each other as effective space-time geometries where Gordon's metric (20) characterizes the geometry seen by the electromagnetic field and the atomic metric (8) describes the effective geometry of the quantum dielectric.

III. MINKOWSKI TENSOR

In order to find the energy-momentum tensor for the dielectric, consider the tensor that would represent the energy momentum of the dielectric matter if the effective metric $g_A^{\alpha\beta}$ describes the true space-time geometry

$$\Theta_\nu^\alpha = mc^2 |\psi|^2 w^{(\alpha)} w_\nu. \quad (22)$$

We obtain from the equations of motion (13) and (15)

$$\begin{aligned} \partial_\alpha \sqrt{-g_A} \Theta_\nu^\alpha &= \sqrt{-g_A} \frac{mc^2}{2} |\psi|^2 w^{(\alpha)} w^{(\beta)} \partial_\nu g_{\alpha\beta}^A \\ &= \sqrt{-g_A} \frac{1}{2} \Theta_{\beta\beta'}^{\alpha\alpha} \partial_\nu g_{\alpha\beta'}^A, \end{aligned} \quad (23)$$

which corresponds to a conservation law in a curved space-time geometry [10] where the right-hand side accounts for the inertial forces. In our case, these are the forces that light exerts on the dielectric medium.

In the following we show that the dielectric forces stem from the Minkowski energy-momentum tensor of the electromagnetic field [2]. The Minkowski tensor is defined as [13]

$$T_{\text{Mkv}}^\alpha = H^{\alpha\alpha'} F_{\alpha'\nu} + \frac{1}{4} H^{\alpha'\beta'} F_{\alpha'\beta'} \delta_\nu^\alpha. \quad (24)$$

We obtain from Maxwell's Eqs. (18)

$$\partial_\alpha T_{\text{Mkv}}^\alpha = H^{\alpha\beta} \partial_\alpha F_{\beta\nu} + \frac{1}{4} \partial_\nu H^{\alpha\beta} F_{\alpha\beta}. \quad (25)$$

We apply the Bianchi identity [10] of the field-strength tensor

$$\partial_\nu F_{\alpha\beta} + \partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha} = 0, \quad (26)$$

that implies

$$H^{\alpha\beta} (2\partial_\alpha F_{\beta\nu} + \partial_\nu F_{\alpha\beta}) = 0. \quad (27)$$

We use Gordon's metric form of the constitutive Eqs. (19) and (21) for ϵ and μ to obtain

$$\begin{aligned} \partial_\alpha T_{\text{Mkv}}^\alpha &= \frac{1}{4} F_{\alpha'\beta'} F_{\alpha\beta} \partial_\nu \frac{\epsilon_0}{\mu} g_F^{\alpha\alpha'} g_F^{\beta\beta'} \\ &= \frac{\epsilon_0}{2} F_{\alpha'\beta'} F_{\alpha\beta} \partial_\nu \left(\epsilon - \frac{1}{\mu} \right) g^{\alpha\alpha'} u^\beta u^{\beta'} - \mathcal{L}_F \partial_\nu \frac{1}{\mu} \\ &= \frac{\epsilon_0}{2} mc^2 b F_{\alpha'\beta'} F_{\alpha\beta} g^{\alpha\alpha'} \partial_\nu w \varrho u^\beta u^{\beta'} - \mathcal{L}_F \partial_\nu \frac{1}{\mu} \\ &= -\frac{mc^2}{2} b (T_{\beta\beta'}^F + \mathcal{L}_F g_{\beta\beta'}) \partial_\nu w \varrho u^\beta u^{\beta'} - \mathcal{L}_F \partial_\nu \frac{1}{\mu}. \end{aligned} \quad (28)$$

We use the relation

$$\partial_\nu \frac{1}{\mu} = \frac{mc^2}{2} (a-b) g_{\beta\beta'} \partial_\nu w \varrho u^\beta u^{\beta'} \quad (29)$$

and the definition (8) of $g_A^{\alpha\beta}$ to arrive at the expressions

$$\begin{aligned} \partial_\alpha T_{\text{Mkv}}^\alpha &= \frac{mc^2}{2} (g_{\alpha\beta}^A - g_{\alpha\beta}) \partial_\nu w \varrho u^\alpha u^\beta \\ &= -\frac{mc^2}{2} w \varrho u^\alpha u^\beta \partial_\nu g_{\alpha\beta}^A + \partial_\nu p. \end{aligned} \quad (30)$$

Here p denotes the dielectric pressure [13]

$$p = \frac{mc^2}{2} (g_{\alpha\beta}^A - g_{\alpha\beta}) w \varrho u^\alpha u^\beta. \quad (31)$$

Using the definition (8) of $g_A^{\alpha\beta}$ and Eqs. (19)–(21) one can express the pressure p in terms of the electromagnetic fields [13]

$$p = \frac{1}{4} F_{\alpha\beta} (H^{\alpha\beta} - \epsilon_0 F^{\alpha\beta}), \quad (32)$$

where in the nonrelativistic limit p approaches the total dipole potential $-(\varrho/2)(\alpha_E E^2 + \alpha_B c^2 B^2)$. Using the definition (22) of Θ_ν^α and the quasiconservation law (23) we obtain

$$\partial_\alpha T_{\text{Mkv}}^\alpha = -\partial_\alpha \sqrt{-g_A} \Theta_\nu^\alpha + \partial_\nu p. \quad (33)$$

This genuine conservation law suggests that the total energy-momentum tensor T_ν^α of light and matter is

$$T_\nu^\alpha = \sqrt{-g_A} \Theta_\nu^\alpha + T_{\text{Mkv}}^\alpha - p \delta_\nu^\alpha \quad (34)$$

that contains the Minkowski tensor (24) as a major building block. We use Eqs. (14), (16), (17), (22), and (32) to arrive at the expression

$$T_\nu^\alpha = \varrho \hbar (\partial_\nu S) c u^\alpha + H^{\alpha\alpha'} F_{\alpha'\nu} + \frac{\varepsilon_0}{4} F_{\alpha'\beta'} F^{\alpha'\beta'} \delta_\nu^\alpha. \quad (35)$$

The components T_l^0 with $l \in \{1, 2, 3\}$ constitute the momentum density \mathbf{g} . We obtain

$$\mathbf{g} = \varrho u^0 \hbar \nabla S + \mathbf{D} \times \mathbf{B}, \quad (36)$$

which agrees with Eq. (1), apart from the factor of $u^0 = (1 - u^2/c^2)^{-1/2}$ that approaches unity in the nonrelativistic limit.

IV. ABRAHAM TENSOR

Formulas (1)–(3) illustrate the ambiguity of the electromagnetic momentum in media and the connection between the Abraham and Minkowski momenta. We expect that the fully relativistic Abraham and Minkowski tensors are connected in a similar way. In order to show this, consider the contravariant tensor

$$\begin{aligned} \sqrt{-g_A} \Theta^{\alpha\nu} &= \sqrt{-g_A} \Theta_{\nu'}^\alpha g^{\nu\nu'} = \sqrt{-g_A} m c^2 |\psi|^2 w^2 u^\alpha u^\beta g_{\beta\nu'}^A g^{\nu\nu'} \\ &= m c^2 \varrho w g_{\beta\nu'}^A g^{\nu\nu'} u^\alpha u^\beta. \end{aligned} \quad (37)$$

According to definition (8) of the atomic metric, $m c^2 \varrho w g_{\beta\nu'}^A$ depends on the free-field Lagrangian and the free-energy-momentum tensor (9) with the coefficients a and b that we express in terms (21) of the permittivity ε and the permeability μ . In addition, we use

$$\begin{aligned} \left(2 - \frac{2}{\mu}\right) \mathcal{L}_F &= 2p - \frac{2}{\mu} \mathcal{L}_F - \frac{1}{2} F_{\alpha\beta} H^{\alpha\beta} \\ &= 2p - \left(\varepsilon - \frac{1}{\mu}\right) \varepsilon_0 F_{\alpha\alpha'} F_{\beta\beta'} u^{\alpha'} u^{\beta'} g^{\alpha\beta}, \end{aligned} \quad (38)$$

and obtain the result

$$\sqrt{-g_A} \Theta^{\alpha\nu} = [(m c^2 w \varrho + 2p) u^\nu - (\varepsilon \mu - 1) \Omega^\nu] u^\alpha. \quad (39)$$

Here Ω^ν abbreviates

$$\begin{aligned} \Omega^\nu &= \frac{\varepsilon_0}{\mu} (u^\nu F_{\alpha\alpha'} F_{\beta\beta'} u^{\alpha'} u^{\beta'} g^{\alpha\beta} + F_{\beta'\beta} F^{\nu\beta'} u^\beta) \\ &= \frac{\varepsilon_0}{\mu} F_{\alpha\alpha'} u^{\alpha'} (u^\nu g^{\alpha\gamma} g^{\beta\beta'} F_{\gamma\beta'} u_\beta + g^{\nu\nu'} g^{\alpha\gamma} F_{\nu'\gamma}) \\ &= \frac{\varepsilon_0}{\mu} F_{\alpha\alpha'} u^{\alpha'} (u^\nu g_F^{\alpha\gamma} g_F^{\beta\beta'} F_{\gamma\beta'} u_\beta + g_F^{\nu\nu'} g_F^{\alpha\gamma} F_{\nu'\gamma}) \\ &= F_{\alpha\alpha'} u^{\alpha'} (u^\nu H^{\alpha\beta} u_\beta + H^{\nu\alpha}) \\ &= F_{\alpha\alpha'} u^{\alpha'} u_\beta (H^{\alpha\beta} u^\nu + H^{\beta\nu} u^\alpha + H^{\nu\alpha} u^\beta) \end{aligned} \quad (40)$$

that is known as Abraham's *Ruhstrahl* [3]. Finally, we ex-

press the pressure (31) in terms of the norm w using the Hamilton-Jacobi Eq. (12)

$$p = \frac{m c^2 \varrho}{2} \left(\frac{1}{w} - w \right), \quad (41)$$

invert this relationship

$$m c^2 w \varrho = \sqrt{m^2 c^4 \varrho^2 + p^2} - p \approx m c^2 \varrho - p, \quad (42)$$

and arrive at the representation

$$\sqrt{-g_A} \Theta^{\alpha\nu} = [(m c^2 \varrho + p) u^\nu - (\varepsilon \mu - 1) \Omega^\nu] u^\alpha. \quad (43)$$

In this way we obtain for the total energy-momentum tensor (34) the expression

$$T^{\alpha\beta} = T_{Ab}^{\alpha\beta} + (m c^2 \varrho + p) u^\alpha u^\beta - p g^{\alpha\beta} \quad (44)$$

that describes the energy-momentum balance between a fluid [15] under the dielectric pressure p and the electromagnetic field in terms of the Abraham tensor [3,13]

$$T_{Ab}^{\alpha\beta} = T_{Mk}^{\alpha\beta} - (\varepsilon \mu - 1) u^\alpha \Omega^\beta. \quad (45)$$

V. CONCLUSIONS

We derived two different forms of the energy-momentum balance in quantum dielectrics: The total energy-momentum tensor can be divided into the Minkowski tensor (24) and the pseudoenergy-momentum tensor Θ_ν^α of the dielectric, apart from a pressure term, or, alternatively, into the Abraham tensor (45) and the energy momentum of a fluid under the dielectric pressure (31). The momentum components of Θ_ν^α depend on the phase of the quantum dielectric, whereas the velocity \mathbf{u} enters the fluid-mechanical component of the total energy-momentum tensor. Due to the Röntgen interaction in dielectric media [8] the momentum derived from the gradient of the phase differs from the mechanical momentum $m\mathbf{u}$, a difference that is usually small and often negligible, whereas the Abraham and Minkowski momenta differ by the refractive index squared. We thus arrive at the remarkable conclusion that the Abraham-Minkowski controversy [4] has its root in the Röntgen interaction [8].

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