The Lorentz force and the radiation pressure of light

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To make plausible the idea that light exerts a pressure on matter, some introductory physics texts consider the force exerted by an electromagnetic wave on an electron. The argument as presented is mathematically incorrect and has several serious conceptual difficulties without obvious resolution at the classical, yet alone introductory, level. We discuss these difficulties and propose an alternative argument.
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I. THE FRESHMAN ARGUMENT

The interaction of light and matter plays a central role, not only in physics itself, but in any introductory electricity and magnetism course. To develop this topic, most courses introduce the Lorentz force law, which gives the electromagnetic force acting on a charged particle, and later discuss Maxwell's equations. Students are then persuaded that Maxwell's equations admit wave solutions that travel at the speed of light, thus establishing the connection between light and electromagnetic waves. At this point instructors generally state that electromagnetic waves carry momentum in the direction of propagation via the Poynting flux vector and that light therefore exerts a radiation pressure on matter. The assertion is not controversial: Maxwell[1] recognized that light should cause a radiation pressure, but his demonstration is not immediately transparent to present-day students.

At least two texts, the *Berkeley Physics Course*[2] and Tipler and Mosca's *Physics for Scientists and Engineers*,[3] attempt to make the suggestion that light carries momentum more plausible by calculating the Lorentz force exerted by an electromagnetic wave on an electron. In their discussion the authors claim—with differing degrees of rigor—to show that a light wave exerts an average force on the electron in the direction of propagation. Tipler and Mosca, for example, then derive an expression for the radiation pressure produced by a light wave.[3] A cursory look at their argument shows that it is incorrect in several obvious ways and in other respects leads rapidly into deep waters.

The purpose of this note is threefold: to demonstrate why the "freshman" argument is incorrect, to show that nontrivial physics must be introduced to correct it, and to present a more plausible derivation of radiation pressure that is accessible to first-year students. Consider, then, the situation shown in Fig. 1. We assume that a light wave propagates in the $+z$-direction, its E-field oscillates in the x-direction, and its B-field oscillates in the y-direction. The wave impinges on a stationary particle with charge q, exerting on it a force according to the Lorentz force law. In units with $c=1$ the Lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1}$$

which reduces to

$$\mathbf{F} = q(E_x - v_z B_y)\hat{\boldsymbol{i}} + q v_x B_y \hat{\mathbf{k}}. \tag{2}$$

The freshman argument goes like this: Assume that $E \sim \sin(\omega t)$ and $B \sim \sin(\omega t)$. The particle is initially accelerated by the E-field in the $+x$-direction and acquires a velocity $v_x > 0$. The magnetic field then exerts a force on the charge equal to $q\mathbf{v} \times \mathbf{B}$, which points in the $+z$-direction, the direction of propagation of the wave. The electromagnetic wave therefore carries a momentum in this direction. Reference 2 states that "…the motion of the charge is mainly due to $\mathbf{E}$. Thus $\mathbf{v}$ is along $\mathbf{E}$ and reverses direction at the same rate that $\mathbf{E}$ reverses direction. But $\mathbf{B}$ reverses whenever $\mathbf{E}$ reverses. Thus $\mathbf{v} \times \mathbf{B}$ always has the same sign."[4]

A moment's reflection shows that the last assertion is false. After one-half cycle, both $\mathbf{E}$ and $\mathbf{B}$ change sign. But because during this time the E-field has accelerated the charge entirely in the $+x$-direction, the electron at that point still has a positive x-velocity. (In other words, the velocity and acceleration are essentially 90° out of phase, as in a harmonic oscillator.) A similar argument holds for the z-velocity. Thus the cross product $\mathbf{v} \times \mathbf{B}$ reverses sign and now points in the negative z-direction, opposite to the direction of propagation. Furthermore, because there is an x-component to the force, one needs to argue that on average it is zero.

Reference 2 claims that the first two terms in Eq. (2) average to zero, the first because $\mathbf{E}$ varies sinusoidally, the second because $\mathbf{B}$ varies sinusoidally as well and because we "…can assume that the increment of velocity along z during one cycle is negligible, that is, we can take the slowly increasing velocity v_z to be constant during one cycle."[4] With these assumptions Ref. 2 concludes that the average force on the charge is $\langle F \rangle = q\langle v_x B_y \rangle \hat{\mathbf{k}}$. Although the result might seem plausible, it is also incorrect because the velocity and magnetic field are out of phase and consequently the time average of their product vanishes. That this is so, as well as the previous claim, can be seen by an integration of the equation of motion.

II. EQUATION OF MOTION

To determine the momentum of the charge, which we take to be an electron, assume the electric and magnetic fields of the light wave are given by $\mathbf{E}=E_0 \sin(\omega t+\phi)\hat{\boldsymbol{i}}$ and $\mathbf{B}=B_0 \sin(\omega t+\phi)\hat{\boldsymbol{j}}$, where ϕ is an arbitrary phase angle. In our units $E_0=B_0$. We set $\mathbf{F}_{\text{Lorentz}}=m d\mathbf{v}/dt$ in Eq. (2) to obtain a pair of coupled ordinary linear first-order equations for the electron velocity:

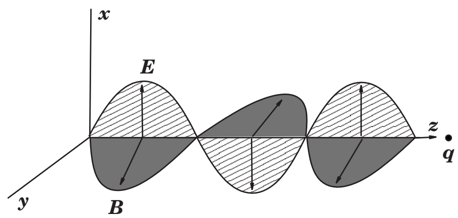

Fig. 1. An electromagnetic wave traveling in the z-direction strikes a point particle with charge q. The E-field is taken in the x-direction and the B-field is taken in the y-direction.

$$\frac{dv_z}{dt} = \omega_c \sin(\omega t + \phi) v_x, \tag{3a}$$

$$\frac{dv_x}{dt} = \omega_c \sin(\omega t + \phi)[1 - v_z], \tag{3b}$$

where we have let $\omega_c \equiv qB_0/m$, the cyclotron frequency.

Equations (3) have the somewhat surprising analytic solutions

$$v_z(t) = c_1 \cos\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] + c_2 \sin\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] + 1, \tag{4a}$$

$$v_x(t) = c_1 \sin\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] - c_2 \cos\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right], \tag{4b}$$

where c_1 and c_2 are the integration constants.

If we take $v_z(0)=v_x(0)=0$, which is reasonable and of sufficient generality for our purposes, we find

$$c_1 = -\cos\left[\frac{\omega_c}{\omega}\cos\phi\right], \quad c_2 = -\sin\left[\frac{\omega_c}{\omega}\cos\phi\right], \tag{5}$$

and the full solutions are therefore

$$v_z(t) = -\cos\left(\frac{\omega_c}{\omega}\cos\phi\right)\cos\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] - \sin\left(\frac{\omega_c}{\omega}\cos\phi\right)\sin\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] + 1, \tag{6a}$$

$$v_x(t) = -\cos\left(\frac{\omega_c}{\omega}\cos\phi\right)\sin\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right] + \sin\left(\frac{\omega_c}{\omega}\cos\phi\right)\cos\left[\frac{\omega_c}{\omega}\cos(\omega t + \phi)\right]. \tag{6b}$$

The behavior of these solutions is not particularly transparent, but can easily be plotted. In Figs. 2–4 we show several graphs for various values of ω_c/ω and phase angle ϕ. Note that regardless of ϕ, v_z is always positive, but that there is also a nonzero v_x whose average can be positive, negative, or zero depending on ϕ. The $\phi=0$ case is shown in Figs. 2 and 4 and the $\phi=\pi/2$ case in Fig. 3. Also, for $v_x \ll 1, v_x \gg v_z$.

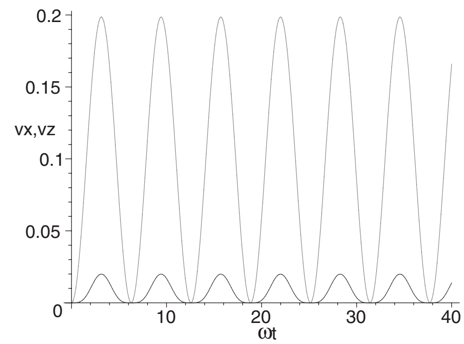

Fig. 2. The x- and z-velocities versus ωt for $\omega_c/\omega=0.1$ and $\phi=0$. Note that from the small ω_c/ω approximation (see text) $v_x \gg v_z$; in this case $v_z/v_x = 0.1$.

Additional insight into the solutions can be obtained by examining the limit $\omega_c/\omega \ll 1$. For ordinary light sources at optical frequencies $\omega \sim 10^{16}$ rad/s, consideration of the Poynting flux (in the following section) gives $\omega_c/\omega \sim 10^{-11}$, and so the limit is well satisfied. For high-powered lasers, such as those at the National Ignition Facility with pulse energy ~2 MJ, it is possible that ω_c exceeds ω. For $\omega_c \ll \omega$, expansion of the solutions (6) to lowest order in ω_c/ω for $\phi=0$ yields

$$v_z \cong \frac{1}{2}\left(\frac{\omega_c}{\omega}\right)^2 [\cos(\omega t) - 1]^2, \tag{7a}$$

$$v_x \cong \left(\frac{\omega_c}{\omega}\right)[1 - \cos(\omega t)]. \tag{7b}$$

Both v_z and v_x are positive definite, as shown in Fig. 2. Therefore their averages must be as well. This in itself contradicts the arguments of Ref. 2 that $\langle F_x \rangle = 0$ but that $\langle F_z \rangle \neq 0$. Note also that v_z is of order $(\omega_c/\omega)^2$, and v_x is of order ω_c/ω. The behavior coincides with the plots, but suggests that because $v_x^2 \sim (\omega_c/\omega)^2$, a consistent, relativistic calculation will significantly change v_z.[5] Moreover, the time averages of both $\dot{v}_x$ and $\dot{v}_z$ vanish to all orders, and so it is in fact impossible to exert a net force on the particle.

One might object to the arguments of this section on the grounds that we have taken **E** and **B** to be simple harmonic $\sim\sin(\omega t)$ rather than wavelike $\sim\sin(kz - \omega t)$. However, it is evident from Eq. (7) that $kz \ll \omega t$ always and that such corrections are therefore negligible, an assertion borne out by numerical calculations.

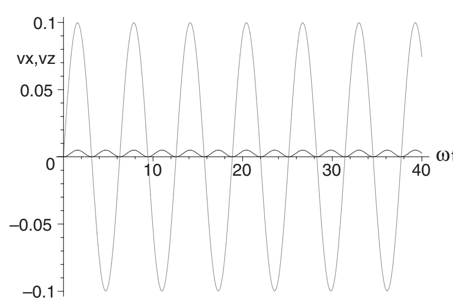

Fig. 3. The same plot as in Fig. 2 except that $\phi=\pi/2$. In this case the time average of $v_x=0$.

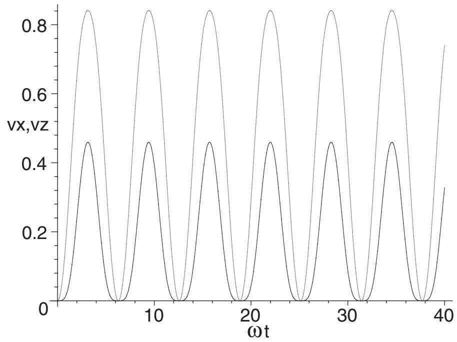

Fig. 4. The same as Fig. 2 except that $\omega_c/\omega=0.5$.

III. INTERPRETATION

The question is whether the behavior just discussed can be reconciled with the classical picture of the Poynting flux. The Poynting vector in our units is

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi}, \tag{8}$$

and the time average is $\langle\mathbf{S}\rangle = \mathrm{R}e(\mathbf{E}\times\mathbf{B}^*)/8\pi$. $\mathbf{S}$ points in the direction of propagation of the electromagnetic wave and in units with $c=1$ can be regarded interchangeably as power per unit area, energy per unit volume (or pressure), or momentum flux. If the freshman argument is correct, then the particle should be accelerated in the direction of the Ponyting vector. But our previous results show that, on the contrary, the particle drifts off in some other direction at a constant average velocity.

Unfortunately, there seem to be several deep inconsistencies in the entire approach. One is that the freshman argument is an invalid attempt to apply the standard classical derivation that is invoked to identify the Poynting flux with electromagnetic momentum, a derivation which breaks down in the limit we have been considering. That is, advanced texts such as Jackson,[6] typically begin by considering the Lorentz force on a volume of charges:

$$\frac{d\mathbf{p}}{dt} = \int_{\mathrm{vol}} (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})d^3x. \tag{9}$$

The first step is to eliminate the charge density ρ in favor of $\mathbf{E}$ via Gauss's law, $\rho=(1/4\pi)\boldsymbol{\nabla}\cdot\mathbf{E}$. We also eliminate $\mathbf{J}$ in favor of $\nabla\times\mathbf{B}$ via Ampère's law to find

$$\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} + \frac{d}{dt}\int_{\mathrm{vol}} \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})d^3x = \frac{1}{4\pi}\int_{\mathrm{vol}} [\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})]d^3x. \tag{10}$$

This result is purely formal, which after the elimination of ρ and $\mathbf{J}$ relies only on vector identities. Because the second term on the left is the only electromagnetic term with a time derivative, we tentatively identify it with the momentum of the field.

The crucial difference between the "graduate" approach and the freshman method, however, is that in the graduate approach we consider a continuous charge distribution. In the limit of a single charge, the ρ in the Lorentz force law becomes the test charge distribution, whereas the ρ in Gauss's law becomes the source charge distribution and they cannot be equated. In the present situation there is not only a single test charge but no source charges whatsoever. Thus the standard derivation cannot be applied. The only volume we have at our disposal is the volume of the electron itself, which leads quickly into quantum territory.

A second difficulty is that the assumption of plane waves with constant amplitude is an assumption of constant energy and momentum. If the light wave has constant momentum, how can any be transferred to the electron? There are many instances in physics where we ignore the backreaction of a recoiling particle on the system. For instance, according to conservation of momentum, a ball should not bounce off a wall, until it is realized that the ball's change in momentum is absorbed by Earth.

Holding the amplitude constant in the current calculation might seem a reasonable approximation, but to be totally consistent we should take into consideration the fact that the electron is accelerating and consequently emits radiation, and with that radiation momentum. The customary way to do this calculation in the nonrelativistic limit is via Thomson scattering. The differential Thomson scattering cross section for a wave polarized in the x-direction is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{m}\right)^2(\cos^2\theta\cos^2\phi + \sin^2\phi), \tag{11}$$

where θ is the angle between the incident and scattered wave. The differential scattering cross section is defined as the ratio of the radiated power per unit solid angle to the incident power per unit area. The Thomson cross section, though, is symmetric with respect to reflection through the origin and consequently as much momentum is emitted in the forward as in the backward direction and hence no net force is exerted on electron. It is therefore not obvious how to remedy this situation in the classical limit. Only when we do a quantum mechanical derivation (Compton scattering) do we see an asymmetry in the scattering cross section. In the present circumstance, however, $\hbar\omega/m_e \sim 10^{-5}$, so it would appear that quantum corrections should be unnecessary.

What we do in practice to obtain the radiation pressure of light in, say, astrophysical calculations is to multiply the time-averaged Poynting flux $\langle\mathbf{S}\rangle$ by the total Thomson cross section σ_T. One can see why this works as follows. A photon scattered off an electron will have a z-momentum $p_z=p_0\cos\theta$ for initial momentum p_0, and it therefore removes $(1-\cos\theta)p_0$ from the original momentum component; the electron must gain the same amount. Multiplying the differential Thomson scattering cross section in Eq. (11) by $(1-\cos\theta)$ and integrating over the sphere gives the total Thomson scattering cross section

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m}\right)^2. \tag{12}$$

Multiplication by the momentum flux of photons will give the total force on the electron. Because the Poynting flux is the momentum flux of photons, the same numerical result is obtained by multiplying the Thomson cross section by the time-averaged Poynting flux. This argument, however, relies on the quantum nature of photons. The Thomson cross section is the nonrelativistic limit of a cross section that must ultimately be derived from QED, and so we see that the freshman argument leads quickly to a situation that might have no resolution in the realm of classical physics!

The failing of Thomson scattering is due to the fact that no energy is removed from the original beam. A possible classical "out" to this situation is to assert that the energy radiated by the electron must be that lost by the incoming beam. Therefore because $E=p$ for a classical wave, momentum conservation implies that the electron must acquire a z-momentum as for the Compton scattering case just discussed.[7] Although this argument is valid in terms of conservation laws, it gives no mechanism for transferring the energy from the incident wave to the electron. Unfortunately, modeling the process as interference between the incident plane wave and the spherical wave outgoing from the electron fails to result in any transfer of z-momentum from the wave to the charge. To recover the Compton result eventually requires including the radiation-reaction force on the electron, which we now consider, but because this derivation involves the classical radius of the electron, it has already gone beyond the realm of classical electromagnetism.

The most straightforward way to deal with the failure of the classical approaches is via the Abraham-Lorentz model, which accounts for the energy radiated by the electron, if in a somewhat *ad hoc* manner. From the Larmor formula the energy radiated by an accelerated electron over a time T is $\sim 2e^2a^2T/3$. Equating this energy to the kinetic energy lost by the particle $\sim ma^2T^2$ gives a characteristic time to lose all the energy to radiation:

$$\tau = \frac{2e^2}{3m}. \tag{13}$$

This timescale is 2/3 the time for light to cross the classical radius of the electron, $r_c = e^2/m$, and has a value $\tau \approx 10^{-23}$ s. The total force acting on a particle is now $m\mathbf{v} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{rad}}$, where $\mathbf{F}_{\text{rad}}$ is the radiation-reaction force. Conservation of energy considerations led Abraham and Lorentz to propose that $\mathbf{F}_{\text{rad}} = m\tau\ddot{\mathbf{v}}$ (see Ref. 7 for more details) and consequently obtain the famous formula

$$m(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{F}_{\text{ext}}. \tag{14}$$

With sufficient massaging, Eq. (14) can be applied to the present circumstance to obtain the desired answer, that is, the force imparted to the electron by an electromagnetic wave is $\mathbf{F} = \langle \mathbf{S} \rangle \sigma_T$. Equation (2) now becomes

$$\dot{v}_x - \tau\ddot{v}_x = \frac{e}{m}(E_x - v_z B_y), \tag{15a}$$

$$\dot{v}_z - \tau\ddot{v}_z = \frac{e}{m} v_x B_y. \tag{15b}$$

In the nonrelativistic regime $v_z \ll 1$ and we ignore the second term on the right in Eq. (15a). We also take both v_x and v_z to be of the form $v = v_0 e^{-i\omega t}$, which is of course manifestly untrue according to the results of Sec. II. Then $\dot{v}_x = -i\omega v_x$ and $\ddot{v}_x = -\omega^2 v_x$. Equation (15a) becomes

$$-i\omega v_x(1 + i\omega\tau) \cong \frac{e}{m}E_x, \tag{16}$$

or with $\omega\tau \ll 1$

$$v_x \cong \frac{ie}{m\omega}E_x(1 - i\omega\tau). \tag{17}$$

With the assumption that $\omega_c/\omega \ll 1$ and $\omega\tau \ll 1$ the $\ddot{v}_z$ term in Eq. (15b) can be ignored. Then

$$\dot{v}_z \cong \frac{ie^2}{m^2\omega}E_x B_y(1 - i\omega\tau). \tag{18}$$

For simplicity, take E_x and B_y to be real. We want the time average of the real part of this expression, or

$$\langle F_z \rangle = \langle m\dot{v}_z \rangle = \frac{e^4}{m^2}\frac{E_0 B_0}{3} = \langle S \rangle \sigma_T. \tag{19}$$

The earliest paper we have found that proposes this calculation is by Page in 1920,[8] although one suspects that Eddington carried it out earlier. Clearly there are a few things to be desired in the derivation, but it does show that the radiation-reaction force is necessary to obtain the claimed result.

With slightly more work the conclusion can be put on a firmer footing via a perturbation calculation[9] as follows. Note that Eq. (7) is the zeroth-order solution of Eq. (15), that is, when $\tau = 0$ and $v_z \ll 1$ is neglected. Assume $v_x \equiv v_{x0} + v_{x1}$ and $v_z \equiv (v_{z0} + v_{z1}) \ll v_x$, where the subscript 0 refers to the zeroth-order solution and the subscript 1 refers to the perturbation. It is then not too difficult to show that the surviving $\dot{v}_z$ is the perturbative part:

$$\dot{v}_{z1} \cong \omega_c v_{x1} \sin(\omega t) \cong \omega_c^2 \tau \sin^2(\omega t). \tag{20}$$

Taking the time average of this expression vindicates the previous result. We emphasize that the Abraham-Lorentz model includes an explicit statement about the structure of the electron and hence cannot be regarded as entirely classical; the model is a transition to quantum mechanics and quantum field theory.

IV. ALTERNATIVE APPROACH

Despite the many pitfalls revealed by the above methods, there is a superior and convincing demonstration that light exerts a pressure on matter, one that should be accessible to students who have had a basic exposure to Maxwell's equations. The advantage of this demonstration is that it avoids consideration of the force acting on a point charge and can therefore be carried out at the purely classical level. For this reason it should be adopted by introductory textbook authors. What follows is a simplified version of a calculation described by Planck.[10]

As before, consider a light wave propagating in the $+z$-direction that bounces off a mirror at $z=0$ (see Fig. 5). We take the mirror to be a near perfect conductor of height dx, width dy, and thickness z. The electric field of the light is a superposition of right- and left-traveling waves:

$$E_x = E_0 \cos(kz - \omega t) - E_0 \cos(kz + \omega t), \tag{21}$$

where $k = 2\pi/\ell$ is the wave number, and we have included a phase change on reflection. (This solution ensures that $E=0$ at the surface of the conductor. Recall that the tangential component of an E-field must be continuous across a boundary, and because the interior field essentially vanishes for a good conductor, the exterior field at the boundary must also.)

From the differential form of Faraday's law,[11] $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, we have

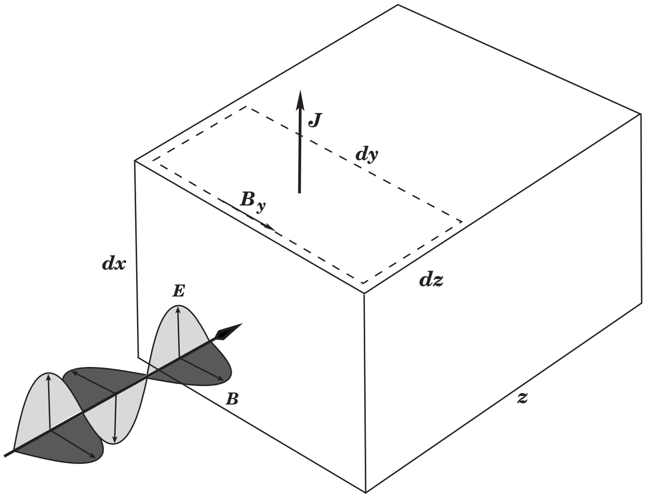

Fig. 5. A light wave traveling in the z-direction strikes an almost perfectly conducting mirror of thickness z, width dy, and height dx. An Ampèrian loop in the yz plane is also shown, with the direction of $\mathbf{B}$ given by the right-hand rule.

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\hat{\boldsymbol{j}} = -E_0 k[\sin(kz-\omega t) - \sin(kz+\omega t)]\hat{\boldsymbol{j}}$$
$$= -\frac{\partial \mathbf{B}}{\partial t}. \tag{22}$$

Integrating with respect to t and remembering that $k=\omega$ in units where $c=1$ gives

$$\mathbf{B} = E_0[\cos(kz-\omega t) + \cos(kz+\omega t)]\hat{\boldsymbol{j}}$$
$$= 2B_0\cos(kz)\cos(\omega t)\hat{\boldsymbol{j}}. \tag{23}$$

Notice that at the boundary, $B=2B_0\cos(\omega t)\neq 0$ and that therefore by Ampère's law, $\oint \mathbf{B}\cdot d\mathbf{s}=4\pi I$, oscillating currents must be induced near the surface of the mirror. Because $\mathbf{B}$ is in the $\pm y$-direction, the right-hand-rule tells us that these currents will be in the $\pm x$-direction, and that $\mathbf{I}\times\mathbf{B}$ will always point in the $+z$-direction. Consequently, the Lorentz force due to the light, $\mathbf{F}=\mathbf{I}dx\times\mathbf{B}$ for a mirror of height dx and total current I, will produce a force in the direction of propagation.

We can calculate the magnitude of the force simply and plausibly. The magnitude of the Lorentz force is $dF=IdxB$, or $dF=JdxdydzB$ for current density J. The differential form of Ampère's law tells us that

$$\nabla \times \mathbf{B} = -\frac{\partial B_y}{\partial z}\hat{\boldsymbol{i}} = 4\pi\mathbf{J}, \tag{24}$$

or $J=-(1/4\pi)\partial B_y/\partial z$. The Lorentz force therefore becomes

$$\frac{dF}{dxdy} = -\frac{1}{4\pi}\frac{\partial B_y}{\partial z}B_y dz. \tag{25}$$

The quantity on the left is dP, where P is the pressure. Because the only spatial dependence of B is on z, we can ignore the distinction between the partial and full differentials. Evidently, because $\partial B_y/\partial z$ is connected to J, we must interpret B as being the field exerting a force on a given slice within the conductor. If we assume that the magnetic field drops off to zero at infinity, which is certainly true inside a good conductor where the falloff is exponential, the total pressure on the mirror should be

$$P = -\frac{1}{4\pi}\int_0^\infty B\,dB = +\frac{1}{8\pi}B(0)^2 = \frac{1}{2\pi}B_0^2\cos^2(\omega t), \tag{26}$$

where the last equality follows from Eq. (23) and the continuity of the tangential component of $\mathbf{B}$ across the boundary. The time average of Eq. (26) gives

$$P = \frac{E_0B_0}{4\pi} = 2\langle S\rangle_{\text{incident}} \tag{27}$$

as desired. Note that the factor of 2 is expected due to the recoil of the wave off the mirror.

There are a few tacit assumptions in this derivation that should be made explicit. One might wonder, for example, why we used Ampère's law (24) to calculate the conduction current, rather than Faraday's law, $d\phi/dt=-\oint\mathbf{E}\cdot d\mathbf{s}=\mathcal{E}$, for the magnetic flux $\phi=Bdxdz$ and the induced EMF $\mathcal{E}$. Normally, we would have students use this law to calculate the induced current $I=\mathcal{E}/R$ in, for example, a wire loop of resistance R. However, in a good conductor $E\ll B$ and hence $|d\phi/dt|=\oint\mathbf{E}\cdot d\mathbf{s}\ll\oint\mathbf{B}\cdot d\mathbf{s}=4\pi I$, the last equality representing Ampère's law.

Furthermore, the B-field in Eq. (24) includes both the incident field and that generated by the induced currents. It seems unreasonable that the portion of the B-field generated by the induced currents can result in a net force on the currents themselves (no "Munchausen effect"[12]). A detailed calculation demonstrates that the integrated force exerted on the induced currents by the induced B-field vanishes. With these assumptions the simpler derivation we have presented is sound and shows that light waves do exert a pressure on matter in the direction of propagation.

In conclusion, although one does not, and cannot, expect derivations at the introductory level to be uniformly rigorous, this case is of particular interest because the interaction of light with matter is of fundamental importance. Moreover, the explanation presented in some textbooks is so seriously flawed that even students sometimes notice the difficulties. Rather than try to paper over these problems with what must be regarded as nonsensical arguments, the occasion would be better exploited to point out that physics is composed of a collection of models that are brought to bear in explaining physical phenomena, but that these models have limited domains of applicability and as often as not are inconsistent.

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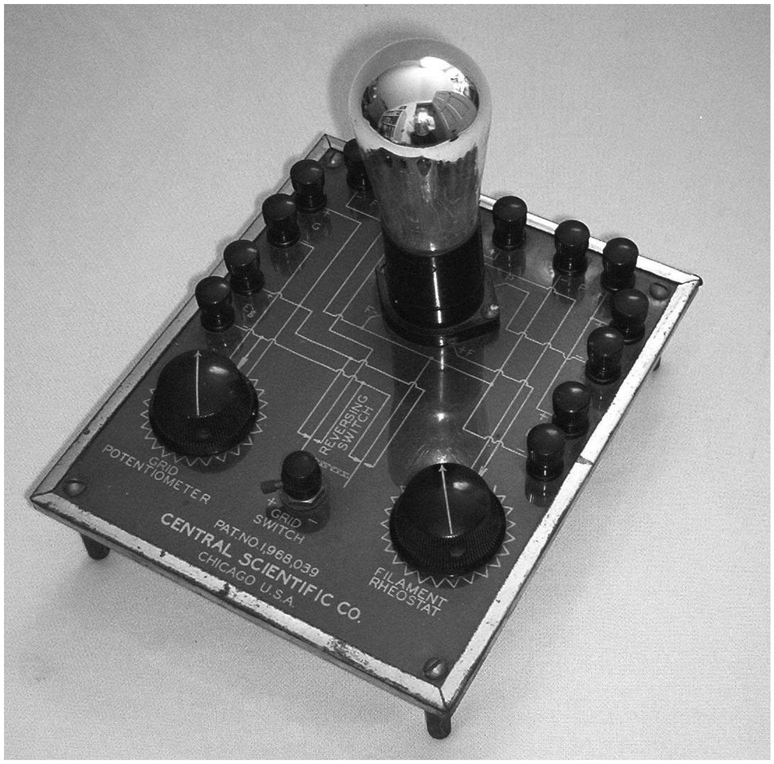
[9] Reference 7, p. 34.

[10] Max Planck, *The Theory of Heat Radiation*, reprint of 1912 edition (Dover, New York, 1991), Part II, Chap. 1.

[11] Most introductory texts use the integral form of Maxwell's equations. The derivation can easily be carried out by considering infinitesimal loops in the xz and yz planes as follows: The integral form of Faraday's law is $\oint \mathbf{E}\cdot \mathbf{ds} = -d\phi/dt$ for magnetic flux ϕ. For the case of our mirror the right-hand rule gives $\oint \mathbf{E}\cdot \mathbf{ds} = E_x(z+dz)dx - E_x(z)dx = (dE_x/dz)dzdx = -(dB_y/dt)dxdz = -(d\phi/dt)$ which leads immediately to Eq. (22). Similarly, the integral form of Ampère's law $\oint \mathbf{B}\cdot \mathbf{ds} = 4\pi I$ leads to Eq. (24).

[12] Anonymous, 1781.



Vacuum Tube Testing Outfit. The 1937 catalogue of the Central Scientific Company of Chicago shows this apparatus at \$14.00 to "aid the elementary student in the usually confusing study of radio vacuum tube characteristics." The student was able to obtain curves of the plate current as a function of the plate voltage for various values of the negative grid voltage. This one, in the Greenslade Collection, has a type 201A triode tube installed. The patent date is from 1935. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)