

LETTER TO THE EDITOR

On the mechanical properties of matter waves**Nicholas K Whitlock, Stephen M Barnett and John Jeffers**

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Abstract

Electromagnetic waves and fluids have locally conserved mechanical properties associated with them and we may expect these to exist for matter waves. We present a semiclassical description of the continuity equations relating to these conserved properties of matter waves and derive a general expression for their respective fluxes.

It has long been known that electromagnetic waves have associated mechanical properties. Maxwell described the radiation pressure exerted by light [1] and Poynting identified the momentum carried by light [2] and that angular momentum was associated with circular polarization [3]. The energy, momentum and angular momentum of electromagnetic waves are particularly important as these quantities are all conserved. Associated with the density of each of these quantities is a flux density (or current density) which is related to the density via continuity equations [4].

The creation of the first Bose–Einstein condensates (BECs) in dilute atomic gases in 1995 [5–7] brought about the possibility of creating coherent matter waves. A BEC in a trap is analogous to an electromagnetic field in a laser cavity, in that macroscopic occupation of a single quantum state occurs. Just as a laser cavity is a source of coherent light, a BEC can be a source of coherent matter waves. To continue this analogy we assert that matter waves will have associated mechanical properties and present a general theory of the local mechanical properties of matter waves. We give, as examples, the densities and flux densities of the matter wave particle number and energy. The same theory applies to their momentum and angular momentum.

The local conservation of particles is a well-established concept in fluid mechanics. It states that in any volume element of a fluid, no matter how small, the rate of change of particles in an element of fluid is equal to the rate at which the particles flow into the surface of the element. This principle is made quantitative in the following equation [8];

$$\frac{\partial}{\partial t} \int_V \rho \, dV = - \oint_S \rho \mathbf{v} \cdot d\mathbf{S}, \quad (1)$$

where ρ is the particle density, \mathbf{v} is the velocity of the fluid and $d\mathbf{S}$ is a surface element with direction normal to the surface enclosing the volume V . If we define the particle current

density (or flux density) $\mathbf{J} = \rho \mathbf{v}$, then we can easily obtain the continuity equation for particles in a fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (2)$$

Similar continuity equations hold for other conserved quantities [8] such as the energy density

$$\frac{\partial w}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{v^2}{2} + \epsilon + \frac{P}{\rho} \right) \right] = 0, \quad (3)$$

where ϵ is the internal energy per unit mass and P is the pressure of the fluid, and the momentum density

$$\frac{\partial p_i}{\partial t} + \frac{\partial}{\partial x_j} (P \delta_{ij} + \rho v_i v_j) = 0. \quad (4)$$

Here the summation convention has been introduced, so the indices (i and j) take the values 1, 2, 3 and a repeated index is understood to be summed. This convention will be assumed for the remainder of the paper.

In a case in which a globally conserved quantity does not obey the continuity equation of the form (2), there will be a source of that quantity. An example of this is a fluid under the influence of an external force, which is a source of momentum. In such a system, it is necessary to modify the equation of continuity to allow for the source term. We would expect that the rate of change of momentum in an element of volume would be equal to the rate at which momentum flows into the surface plus the rate at which momentum is created in the element. This reasoning leads us to a modified continuity equation of the form

$$\frac{\partial p_i}{\partial t} + \frac{\partial}{\partial x_j} \Pi_{ij} = S_i, \quad (5)$$

where Π_{ij} is the $\{i, j\}$ component of the momentum flux density tensor and S_i is the source density for the i component of momentum.

We now turn our attention to the semiclassical description of matter waves. Consider an observable A having a corresponding Hermitian operator \hat{A} . The density $\hat{A}(\mathbf{R})$ of A at a point \mathbf{R} will obey the generalized local continuity equation

$$\frac{\partial}{\partial t} \hat{A}(\mathbf{R}) + \frac{\partial}{\partial X_i} \hat{T}_i^{(A)}(\mathbf{R}) = \hat{S}^{(A)}(\mathbf{R}), \quad (6)$$

where X_i are the components of \mathbf{R} , $\hat{T}_i^{(A)}$ is the flux density of the observable and $\hat{S}^{(A)}$ is the source density term. We define the Hermitian operator associated with the density of A at \mathbf{R} to be

$$\hat{A}(\mathbf{R}) = \frac{1}{2} \{ \hat{A}, \delta(\hat{\mathbf{r}} - \mathbf{R}) \}, \quad (7)$$

where $\{, \}$ denotes the anticommutator and $\delta(\hat{\mathbf{r}} - \mathbf{R})$ is the Dirac delta function. From the Heisenberg equation of motion we can show that the density will evolve according to

$$\frac{\partial}{\partial t} \hat{A}(\mathbf{R}) = \frac{i}{2\hbar} (\{ [H, A], \delta(\hat{\mathbf{r}} - \mathbf{R}) \} + \{ \hat{A}, [H, \delta(\hat{\mathbf{r}} - \mathbf{R})] \}). \quad (8)$$

We see that the first term on the right-hand side will vanish if A is conserved. It is therefore reasonable to associate this term with the source term in (6) and the last term with the flux density. In order to proceed in the calculation we use the standard Hamiltonian for a particle in a potential $V(\mathbf{r})$,

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla_i \nabla_i + V(\mathbf{r}), \quad (9)$$

where $\nabla_i \equiv \frac{\partial}{\partial x_i}$. On evaluating the last term in (8) we find that

$$\frac{1}{2} \frac{\partial}{\partial t} \{ \hat{A}, \delta(\hat{\mathbf{r}} - \mathbf{R}) \} + \frac{1}{4} \nabla_{R,i} \left\{ \hat{A}, \left\{ \frac{\hat{p}_i}{m}, \delta(\hat{\mathbf{r}} - \mathbf{R}) \right\} \right\} = \frac{1}{2} \left\{ \frac{\partial}{\partial t} \hat{A}, \delta(\hat{\mathbf{r}} - \mathbf{R}) \right\}, \quad (10)$$

which takes the form of a conservation equation (6). This allows us to identify the flux density operator

$$\hat{\mathcal{T}}_i^{(A)}(\mathbf{R}) = \frac{1}{4} \left\{ \hat{A}, \left\{ \frac{\hat{p}_i}{m}, \delta(\hat{\mathbf{r}} - \mathbf{R}) \right\} \right\}. \quad (11)$$

We can associate the operator $\{ \hat{p}_i/m, \delta(\hat{\mathbf{r}} - \mathbf{R}) \}$ with the velocity density and hence see that $\hat{\mathcal{T}}_i^{(A)}(\mathbf{R})$ is a suitably symmetrized product of the velocity density and the observable. This operator is the flow of the density of the observable, that is its flux density. The forms of the flux density and source density for the observable A will come from the expectation value of (10). In the Heisenberg picture, the matter wave will be described by a wavefunction $\psi(\mathbf{r})$. The expectation value of the conservation equation for this state is

$$\frac{\partial}{\partial t} \mathcal{A}(\mathbf{R}) + \nabla_{R,i} \mathcal{T}_i^{(A)}(\mathbf{R}) = \mathcal{S}^{(A)}(\mathbf{R}), \quad (12)$$

which is the continuity equation for the expectation values of the relevant operators. Here we have introduced the notation $\mathcal{A}(\mathbf{R}) \equiv \langle \hat{\mathcal{A}}(\mathbf{R}) \rangle$. The flux density expectation value is

$$\begin{aligned} \mathcal{T}_i^{(A)}(\mathbf{R}) &= \int d^3r \psi^*(\mathbf{r}) \hat{\mathcal{T}}_i^{(A)}(\mathbf{R}) \psi(\mathbf{r}) \\ &= \frac{i\hbar}{4m} [\psi \nabla_i (\hat{A} \psi)^* - (\hat{A} \psi)^* \nabla_i \psi + \hat{A} \psi (\nabla_i \psi^*) - \psi^* \nabla_i (\hat{A} \psi)]|_{r=\mathbf{R}} \end{aligned} \quad (13)$$

and the source density expectation value is

$$\mathcal{S}^{(A)}(\mathbf{R}) = \frac{i}{2\hbar} \{ -([\hat{H}, \hat{A}] \psi)^* \psi + \psi^* ([\hat{H}, \hat{A}] \psi) \}|_{r=\mathbf{R}}. \quad (14)$$

As an example we consider two quantities which obey continuity equations, specifically the particle number and energy. Following Landau [9], the operator corresponding to particle density in a semiclassical theory is represented by the Dirac delta function, so in (7) we put the operator \hat{A} equal to the identity operator $\hat{1}$. From (13) the flux density of particles at position \mathbf{r} can be calculated as

$$\mathcal{T}_i^{(\rho)}(\mathbf{R}) = \frac{\hbar}{m} \text{Im} \{ \psi^* \nabla_i \psi \}|_{r=\mathbf{R}}, \quad (15)$$

where Im indicates the imaginary part. This is the form of the well-known quantum mechanical probability flux J_i [10]. The source term vanishes as the identity operator commutes with the Hamiltonian. This is a reasonable result as there is no particle source present in the system. Energy in quantum mechanics is represented by the Hamiltonian operator, so we can calculate the energy flux density from (13). If we use the Hamiltonian (9) we obtain the energy flux density at position \mathbf{R} , which is

$$\mathcal{T}_i^{(E)}(\mathbf{R}) = \frac{\hbar^3}{4m^2} \text{Im} \{ \psi \nabla^2 \nabla_i \psi^* + (\nabla_i \psi^*) \nabla^2 \psi \}|_{r=\mathbf{R}} + V(\mathbf{R}) \mathcal{T}_i^{(\rho)}(\mathbf{R}). \quad (16)$$

The source term vanishes as the Hamiltonian commutes with itself.

In this letter, we have derived general flux density and source density operators for conserved quantities in matter waves within the semiclassical description. This theory will be important when considering the deposition of these quantities during interactions of matter waves with other objects. An example of this is the detection of matter waves themselves, which requires one to consider the particle flux density [11]. One problem is that the flux

density operator, expression (11), might contain an arbitrary divergenceless operator in addition to the terms present. It seems likely, as with Poynting's theorem, that this freedom will have no physical consequences [12]. Also of interest is the form of the second-quantized version of the theory presented here. We will address these issues elsewhere.

Acknowledgments

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References

- [1] Maxwell J C 1998 *A Treatise on Electricity and Magnetism* vol 2 (Oxford: Oxford University Press) p 440
- [2] Poynting J H 1884 *Phil. Trans.* **175** 343
- [3] Poynting J H 1909 *Phil. Trans. R. Soc. A* **83** 560
- [4] Barnett S M 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** S7, and references therein
- [5] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 *Science* **269** 198
- [6] Davis K B, Mewes M-O, Andrews M R, van Druten N J, Durfee D S, Durn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969
- [7] Bradley C C, Sackett C A, Tollett J J and Hulet R R 1995 *Phys. Rev. Lett.* **75** 1687
- [8] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* 2nd edn (Oxford: Butterworth-Heinemann) pp 2, 9
- [9] Landau L 1941 *J. Phys.* **5** 71
- [10] Bransden B H and Joachain C J 1989 *Introduction to Quantum Mechanics* (Essex: Longman Scientific and Technical) p 84
- [11] Whitlock N K, Barnett S M and Jeffers J 2003 *J. Phys. B: At. Mol. Opt. Phys.* **36** 1273
- [12] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley) p 259