unity. Also θ_1 is a primitive in $(c_1 l_1)$-th root of unity. This function also has the same property given in (3) and we may obtain relations analogous to (16) involving the quantities $\{i + h, j + k\}\{i, j\}$. As we were principally interested in the case where $cl = p^t - 1$ in (1), in view of certain applications, we did not carry out this more general investigation in detail.

In a former paper[2] we examined the number (i, j) of solutions g^r, g^s of

$$1 + g^{i+rl} = g^{j+sl},$$

where r and s are each in the range of $0, 1, \ldots, c - 1$ and obtained quadratic relations involving the quantities $(i, j)(j + h, j + k)$, given in Theorem I of that paper. The relations seem to be of somewhat different character from those of the present paper and we do not yet have a unified theory to cover the case where c and l are any integers in (1) of such a nature as to yield as special cases relations (16) of this paper as well as (23), (24), (25), (26) of the former paper.

---

[1] These PROCEEDINGS, **32,** 317 (1946).

[2] *Ibid.,* **33,** 236–242 (1947).

[3] *Ibid.,* **31,** 189 (1945).

A SPECIAL CLASS OF SOLUTIONS OF THE EQUATIONS OF THE GRAVITATIONAL FIELD ARISING FROM CERTAIN GAUGE-INVARIANT ACTION PRINCIPLES

BY H. A. BUCHDAHL

PHYSICS LABORATORY, UNIVERSITY OF TASMANIA

Communicated by Oswald Veblen, December 18, 1947

1. By direct consideration of the variation of the integrals

$$J_1 = \int G^2 \sqrt{-g}\, d\tau, \qquad J_2 = \int G^{\mu\nu} G_{\mu\nu} \sqrt{-g}\, d\tau$$

the author has shown[1] that in a V_4 the field equations arising out of the action principles $\delta J_1 = 0$, $\delta J_2 = 0$ are satisfied by any solution of the equations

$$G_{\mu\nu} = \alpha g_{\mu\nu}, \tag{1}$$

where α is an *arbitrary* constant, and $G_{\mu\nu}$ is the Ricci tensor. Let $\overline{B}_{\mu\nu\alpha}{}^{\beta}$ be the gauge-invariant curvature tensor of Weyl's theory;[2] and let $\overline{B}_{\mu\nu\alpha}{}^{\alpha} = \overline{G}_{\mu\nu}$, $g^{\mu\nu}\overline{G}_{\mu\nu} = \overline{G}$, so that in a region free from electromagnetic radiation $\overline{G}_{\mu\nu} = G_{\mu\nu}$, $\overline{G} = G$. Then by a slight extension of the method referred to above we may prove the following:

THEOREM. *In a region free from radiation* ($\kappa_\mu = 0$) *the field equations arising from a gauge-invariant action principle in which the Lagrangian is composed of the components of the contracted curvature tensor and the components of the metrical tensor are satisfied by any solution of the equations* $G_{\mu\nu} = \alpha g_{\mu\nu}$, *where* α *is an arbitrary constant.*

2. Defining the "absence of radiation" to mean

$$\kappa_\mu = 0, \qquad (\mu = 1, 2, \ldots, n) \tag{2}$$

we may prove the theorem somewhat more generally for a V_n. When (2) applies every action principle of the kind under consideration reduces to the form

$$\delta \int L(K_1, K_2, \ldots, K_m) \sqrt{-g}\, d\tau = 0 \tag{3}$$

where the Lagrangian is some function of the $K_s = G^{\mu_2}_{\mu_1} G^{\mu_3}_{\mu_2} G^{\mu_4}_{\mu_3} \ldots G^{\mu_1}_{\mu_s}$, ($K_1 = G$), and m is a positive integer. Equation (3) is the degenerate form of the action principle $\delta \int \overline{L} \sqrt{-g} d\tau = 0$, where the Lagrangian $\overline{L}$ is composed of the $\overline{G}_{\mu\nu}$ and the $g^{\mu\nu}$. In a gauge transformation in which $g_{\mu\nu}$ becomes $\lambda^2 g_{\mu\nu}$, $\sqrt{-g}$ becomes $\lambda^{+n}\sqrt{-g}$ and $\overline{G}_{\mu\nu}$ remains unchanged, so that $\overline{L}$ must change into $\lambda^{-n}\overline{L}$ since $\overline{L}\sqrt{-g}$ is to be a gauge-invariant density. Putting $\kappa_\mu = 0$ after the transformation has been carried out this implies the condition

$$L(\mu K_1, \mu^2 K_2, \ldots, \mu^m K_m) \equiv \mu^{n/2} L(K_1, K_2, \ldots, K_m), \tag{4}$$

where we have set $\mu = \lambda^{-2}$.

Differentiating (4) with respect to μ and putting $\mu = 1$ after the differentiation, this gives

$$\sum_{s=1}^{m} s K_s \frac{\partial L}{\partial K_s} = {}^1/_2 nL. \tag{5}$$

Now

$$\delta L = \sum_{s=1}^{m} \frac{\partial L}{\partial K_s} \delta K_s; \tag{6}$$

and in view of the symmetry of K_s with respect to the $G^{\mu_{i+1}}_{\mu_i}$ we have

$$\delta K_s = s G^{\mu_2}_{\mu_1} G^{\mu_3}_{\mu_2} \ldots G^{\mu_s}_{\mu_{s-1}} \delta G^{\mu_1}_{\mu_s}.$$

If now $G_{\mu\nu}$ be given by (1) it follows immediately that

$$K_s = \text{const.} = n\alpha^s, \qquad \delta K_s = s\alpha^{s-1}\delta G = (sK_s/n\alpha)\delta G.$$

Hence, from (6),

$$\delta L = \frac{1}{n\alpha} \sum_{s=1}^{m} s K_s \frac{\partial L}{\partial K_s} \delta G = L\delta G/2\alpha,$$

by (5). Therefore.

$$\delta\int L\sqrt{-g}d\tau = \int\left(\frac{1}{2\alpha}L\sqrt{-g}\delta G + L\delta\sqrt{-g}\right)d\tau$$
$$= (L/2\alpha)\int(\delta(\sqrt{-g}G) - G\delta\sqrt{-g} + 2\alpha\delta\sqrt{-g})d\tau,$$

or

$$\delta\int L\sqrt{-g}d\tau = (L/2\alpha)\delta\int(G - (n-2)\alpha)\sqrt{-g}d\tau. \qquad (7)$$

Accordingly the condition that the right-hand side of (7) should vanish is always compatible with equation (1) whereby it was transformed into the condition expressed by the vanishing of the right-hand side of (7). For[3]

$$\delta\int(G - (n-2)\alpha)\sqrt{-g}d\tau = \int({}^1/{}_2 g^{\mu\nu}G - G^{\mu\nu} - {}^1/{}_2(n-2)\alpha g^{\mu\nu})\sqrt{-g}\delta g_{\mu\nu}d\tau = 0,$$

by (1). Hence $\delta\int L\sqrt{-g}d\tau$ vanishes, and the theorem of section 1 is proved.

In particular the result applies to the case where the action is taken to be the generalized volume $\int\sqrt{-\det|\bar{G}_{\mu\nu}|}d\tau$, so that

$$L = \sqrt{\det|G_{\mu\nu}|/g} \equiv (\sum a_{\nu_1}\ldots{}_{\nu_m}K_1^{\nu_1}K_2^{\nu_2}\ldots K_m^{\nu_m})^{1/2}, \qquad (\sum_{r=1}^{m} r\nu_r = n),$$

where the a's are certain numerical coefficients. This clearly is of a form satisfying the condition (5).

---

[1] Buchdahl, H. A., *Proc. Edinburgh Math. Soc.* (in the press) (1948).

[2] Eddington, A. S., *The Mathematical Theory of Relativity*, Cambridge University Press, 1930, p. 204.

[3] Reference 2, p. 139.

PHYSICAL FAMILIES OF CURVES IN SPACE

By Edward Kasner and John De Cicco

Departments of Mathematics, Columbia University, New York, and Illinois Institute of Technology, Chicago

Read before the Academy, November 18, 1947

1. In preceding papers, we studied physical families of curves in the plane. We gave characteristic properties and discussed the transformation theory. In the present paper, we shall generalize this theory to space of three dimensions.

The important physical families of curves connected with an arbitrary (positional) field of force[1] are (1) trajectories, that is, the paths of motion of a particle; (2) brachistochrones, that is, the curves in a general (con-