Angular momentum and the geometrical gauge of localized photon states

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We have examined the physical meaning of the geometric gauge associated with the photon position operator and find that localized photon states are not spherically symmetric and may have orbital angular momentum and optical vortices determined by the choice of gauge.

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I. INTRODUCTION

Recently there has been significant advancement of our understanding of optical angular momentum [1–3]. Orbital angular momentum adds to the photon's well-known spin angular momentum to give total angular momentum. Beams with spin and orbital angular momentum are obtainable using standard lasers. A quarter-wave plate, which provides a relative quarter-cycle phase shift between orthogonal linear polarizations, can transform a linearly polarized beam to one with spin angular momentum. Cylindrical lenses or holograms refract the wave fronts and impart orbital angular momentum to a Hermite-Gaussian beam, converting it to a Laguerre-Gaussian beam with helical wave fronts. An annular slit or a cylindrical prism will produce nondiffracting Bessel beams with orbital angular momentum. A number of experiments have demonstrated the transfer of orbital angular momentum to micron sized particles, proving that photon orbital angular momentum is physically real [4].

Although total angular momentum of the classical field is conserved, $\underline{\mathbf{J}}$ cannot be covariantly separated into the usual spin and orbital angular momentum operators $\underline{\mathbf{S}}$ and $\underline{\mathbf{L}}$ [5]. Here the underscore denotes a matrix. In a second-quantized formalism, transversality is maintained, but the spin and orbital angular momentum operators are not true angular momenta [6]. However, states *can* be constructed that have well-defined total angular momentum per photon in any specified direction. Without loss of generality we can choose this to be the z direction, and describe the total, spin, and orbital angular momenta along this direction with quantum numbers j_z, s_z, and l_z, respectively. For example, azimuthally polarized Bessel beams for which $j_z=0$ are superpositions of two components: one with $l_z=1$ and $s_z=-1$, and one with $l_z=-1$ and $s_z=1$ [7].

In spite of an extensive literature on nonlocalizability of photons in three dimensions, localized photon states with arbitrarily fast asymptotic power-law [8] or exponential [9] falloff of energy density have recently been constructed. What has not been analyzed in the past is the deviation of these localized states from spherical symmetry and their consequent angular momentum content. It has been argued that a converging or diverging one-photon state can never be localized exactly because of mathematical limitations imposed by quantum field theory and, for example, the Paley-Wiener theorem [9]. However, a momentum-space basis of exactly localized states can be constructed [10,11] with contributions from all $\mathbf{p}$, describing a photon that may be incoming or outgoing relative to the spacetime point of localization. This orthogonal basis, while probably not realizable as physical photon states, is convenient for calculation of the probability amplitude for photon position and for the specification of transverse bases in general.

In the present paper we discuss the angular momentum of photons localized in all three spatial dimensions.

II. LOCALIZED BASIS STATES AND GEOMETRIC GAUGE

Basis states of the form

$$\underline{\Psi}_{\mathbf{r}',\lambda}(\mathbf{p}) = Np^{\alpha}e^{-i\mathbf{r}'\cdot\mathbf{p}/\hbar}\underline{e}_{\mathbf{p}\lambda}. \qquad (1)$$

are eigenstates of a Hermitian photon position operator with commuting components. In spite of a long history arguing against the existence of such an operator, we found not one, but a whole family of such position operators related by geometric gauge transformations, with the gauge potential defining the rotation of the transverse basis about $\mathbf{p}$. Details, together with explanations where arguments against their existence fail, are given in Ref. [11]. We show here that the gauge choice determines the angular momentum of the basis.

Massless particles possess only two linearly independent spin states, commonly taken to be eigenstates of the helicity operator $\underline{\mathbf{S}}\cdot\hat{\mathbf{p}}$ The resulting coupling of spin and momentum means that the position operator, which generates translations in momentum space, generally does not commute with $\underline{\mathbf{S}}$. For particles of spin 1, the components of $\underline{\mathbf{S}}$ are represented by 3×3 matrices that generate rotations of the field vectors, and the position operator is therefore not simply $i\hbar\boldsymbol{\nabla}$, where $\boldsymbol{\nabla}$ is the gradient operator in $\mathbf{p}$ space, but rather a 3×3 matrix. An early (1948) proposal for such an operator is the Pryce photon position operator [12],

$$\underline{\mathbf{r}}_P = \hbar\left(i\underline{I}p^{\alpha}\,\boldsymbol{\nabla}\,p^{-\alpha} + \frac{1}{p^2}\mathbf{p}\times\underline{\mathbf{S}}\right), \qquad (2)$$

where $\alpha=\frac{1}{2}$ for electromagnetic fields and $-\frac{1}{2}$ for vector potentials, $\underline{\mathbf{S}}$ is the dimensionless spin-1 operator, and $\underline{I}$ is the unit matrix. In our notation, the underscore denotes an array and boldface denotes a three-component vector. Thus, the boldface signifies that $\underline{\mathbf{r}}_P$ has x, y, and z components, while its underscore means that each of these components is a 3×3 array that operates on the vector field of a first-quantized photon state, expressed as a 3×1 array. This notation is carefully maintained here to prevent confusion between these two vector roles.

The Cartesian components of $\underline{\mathbf{r}}_P$ do not commute and thus cannot define a basis of localized states. A family of position operators that do have commuting components is

$$\underline{\mathbf{r}}^{(\chi)} = i\hbar \underline{D} p^{\alpha} \nabla p^{-\alpha} \underline{D}^{-1} \tag{3}$$

where $\underline{D}=e^{-i\underline{S}_3\phi}e^{-i\underline{S}_2\theta}e^{-i\underline{S}_3\chi}$ is the rotation matrix with Euler angles ϕ,θ,χ that rotates the laboratory z axis into $\hat{\mathbf{p}}$. The role of the matrices $\underline{D}$ and $\underline{D}^{-1}$ is to decouple the spin and momentum, allowing the gradient operator to operate on the momentum dependence of the field while maintaining the transversality condition. A straightforward calculation gives [10,11]

$$\underline{\mathbf{r}}^{(\chi)} = \underline{\mathbf{r}}_P - \hbar \mathbf{a}^{(\chi)} \hat{\mathbf{p}} \cdot \underline{\mathbf{S}} \tag{4}$$

with

$$\mathbf{a}^{(\chi)}(\theta,\phi) = \frac{\cos\theta}{p\sin\theta}\hat{\boldsymbol{\phi}} + \nabla\chi(\theta,\phi). \tag{5}$$

The polar and azimuthal angles are denoted θ and ϕ in momentum space and ϑ and φ in position space.

As the basis vectors for the field and hence for the first-quantized photon wave function, we use complex vectors $\underline{e}_\lambda$ of definite helicity $\lambda=\pm 1$, with components

$$e^{(\chi)}_{\lambda,\mu}(\theta,\phi) = e^{(0)}_{\lambda,\mu}(\theta,\phi)\exp(-i\lambda\chi), \tag{6}$$

where

$$e^{(0)}_{\lambda,\mu}(\theta,\phi) = (\hat{\theta}_\mu + i\lambda\hat{\phi}_\mu)/\sqrt{2}, \tag{7}$$

and we add $e_{0,\mu}=\hat{p}_\mu$ to complete the triad. The caret denotes a unit vector, and $\mu=-1$, 0, and 1 label rows of the column vector $\underline{e}_\lambda$, and denote components on the complex vectors $(\hat{\mathbf{x}}-i\hat{\mathbf{y}})/\sqrt{2}$, $\hat{\mathbf{z}}$, and $(\hat{\mathbf{x}}+i\hat{\mathbf{y}})/\sqrt{2}$, respectively, which are eigenvectors of $\underline{S}_z$ with eigenvalue μ. Here we express the rotation matrix $\underline{D}$ in terms of the same components [13] and note that $e^{(\chi)}_{\lambda,\mu}(\theta,\phi)=D_{\mu\lambda}=D^*_{-\mu,-\lambda}$. The general transverse basis vector $\underline{e}^{(\chi)}_\lambda$ is rotated relative to $\underline{e}^{(0)}_\lambda$ by the Euler angle χ about $\mathbf{p}$, giving just a phase difference in the helicity basis.

While the phase of the basis vectors depends on the choice of χ, the physical fields are obviously independent of how we choose to orient the basis vectors around $\mathbf{p}$. Indeed, we can rotate the basis vectors around $\mathbf{p}$ by a different angle at different momentum-space positions (θ,ϕ), and this cannot change the physical field. In this sense, a reorientation transformation $\chi(\theta,\phi)\to\chi'(\theta,\phi)$ is a true local gauge transformation. It is a basic requirement of the covariance of the geometric representation. The invariance of the physical field and hence the photon wave function means that the coefficients of the field when expanded in the basis receive compensating phase factors [14].

The term $\mathbf{a}^{(\chi)}$ may be considered an Abelian momentum-space vector potential, analogous to the vector potential $\mathbf{A}$ of electromagnetic theory [11]. The position operator $\underline{\mathbf{r}}^{(\chi)}$ in Eq. (4) depends on the gauge of $\mathbf{a}^{(\chi)}$ through $\nabla\chi(\theta,\phi)$, similar to the way the kinetic momentum of a massive charged particle depends on the gauge of $\mathbf{A}$. The role of the charge of the massive particle is seen to be taken in momentum space by the helicity of the photon, and it is relevant to recall here the well-known result that the helicity defines an invariant subspace of the Poincaré group. The basis vectors are taken as eigenstates of the position operator, and a gauge transformation cannot change their eigenvalues. Thus, a gauge transformation in the basis states $\underline{e}^{(\chi)}_\lambda$ of the helicity subspace λ, say $\underline{e}^{(\chi)}_\lambda \to \underline{e}^{(\chi')}_\lambda = T\underline{e}^{(\chi)}_\lambda$, must change the position operator according to the usual gauge rule

$$\mathbf{r}\underline{e}^{(\chi)}_\lambda \to \mathbf{r}'\underline{e}^{(\chi')}_\lambda = T\mathbf{r}\underline{e}^{(\chi)}_\lambda,$$

and this gives the transformation $\mathbf{r}'=T\mathbf{r}T^{-1}$. In our case, T is the phase factor $T=e^{-i\lambda(\chi'-\chi)}$ and $\underline{\mathbf{r}}=i\hbar\underline{D}p^{\alpha}\nabla p^{-\alpha}\underline{D}^{-1}$ so that $\underline{\mathbf{r}}'=\underline{\mathbf{r}}-\lambda\nabla(\chi'-\chi)$, which is exactly the dependence we find for $\underline{\mathbf{r}}$ on the gauge transformation.

The field $\nabla\times\mathbf{a}^{(\chi)}$ in momentum space corresponds to that of a magnetic monopole at the origin. For the potential $\mathbf{a}^{(0)}$, it has singular "Dirac" strings of flux lines on the $\pm z$ axis that supply the flux emanating from the monopole. This is most easily seen by integrating $\mathbf{a}^{(0)}$ along a path encircling the z axis and equating this to the flux passing through the area bounded by the path. The singular strings associated with $\mathbf{a}^{(\chi)}$ represent an essential nonintegrability or path dependence that is responsible for the physical manifestation of the gauge potential [15]. Gauge transformations induced by changes in $\chi(\theta,\phi)$ can change the strings, but they do not alter the physical results. As shown in Ref. [11], the Abelian potential $\mathbf{a}^{(\chi)}$ is part of a more general nonabelian gauge potential for SO(3).

The basis defined by

$$\chi(\theta,\phi) = -m\phi \tag{8}$$

has total z angular momentum quantum number $j_z=m\lambda$ with the single-valued gauge potential

$$\mathbf{a}^{(\chi)} = \hat{\boldsymbol{\phi}}\frac{\cos\theta - m}{p\sin\theta}. \tag{9}$$

The singularities in $\mathbf{a}^{(0)}$ along the $\pm z$ axis $(\theta=0,\pi)$ are thus changed in strength by the factors $1\mp m$. For example, for $m=1$, the singularity along the positive z axis is missing in $\mathbf{a}^{(\chi)}$ whereas that along the negative z axis carries twice the flux. Other choices of $\chi(\theta,\phi)$ can reorient the singularity along some other direction or replace it by a nonintegrable (multivalued) $\mathbf{a}^{(\chi)}$. A reorientation of the singularity does not produce any new physics, and as discussed above, for simplicity we choose a geometric gauge with the singularity on the $\pm z$ axis. (A more general choice of χ can give a singularity that is not straight as discussed in the literature on magnetic monopoles [18], perhaps with interesting consequences.) Restricting the Euler angle χ to functions given by Eq. (8), the basis vectors can be expanded in eigenvectors of the usual spin-1 matrix $\underline{S}_z$ and $L_z=-i\partial/\partial\phi$ as

$$\underline{e}_{\lambda}^{(-m\phi)} = \frac{1}{2}\begin{pmatrix} (\cos\theta - \lambda)\exp[i(m\lambda + 1)\phi] \\ -\sqrt{2}\sin\theta\exp(im\lambda\phi) \\ (\cos\theta + \lambda)\exp[i(m\lambda - 1)\phi] \end{pmatrix}. \quad (10)$$

The top row ($\mu=-1$) gives the projection of the basis state $\underline{e}_{\lambda}^{(-m\phi)}$ onto a state with $\underline{S}_z$ quantum number -1 and L_z quantum number $m\lambda+1$ with probability $\frac{1}{4}(\cos\theta-\lambda)^2$; the second row ($\mu=0$) has the corresponding quatnum numbers 0 and $m\lambda$ with probability $\frac{1}{2}(\sin\theta_{\mathbf{p}})^2$; while the third row ($\mu=1$) has quantum numbers 1 and $m\lambda-1$ with probability $\frac{1}{4}(\cos\theta+\lambda)^2$. Thus, by inspection, it is confirmed that the total angular momentum quantum of $\underline{J}_z$ of the basis state is $\hbar m\lambda$. The expectation values of $\underline{S}_z$ and L_z for the basis state, obtained from the weighted sum, are then $\hbar\cos\theta$ and $\hbar(-\cos\theta+m\lambda)$, respectively, showing that its cosine terms exactly cancel, leaving the eigenvalue $\hbar m\lambda$ of $\underline{J}_z$.

To be consistent with the position operator $\underline{\mathbf{r}}^{(\chi)}$, a modified external angular momentum operator can be introduced as

$$\underline{\mathbf{L}}^{(\chi)} = \underline{\mathbf{r}}^{(\chi)} \times \mathbf{p}. \quad (11)$$

The total angular momentum is then written as $\underline{\mathbf{J}}=\underline{\mathbf{S}}^{(\chi)}+\underline{\mathbf{L}}^{(\chi)}$ where the internal angular momentum is

$$\underline{\mathbf{S}}^{(\chi)} = (\mathbf{a}^{(\chi)} \times \mathbf{p} + \hat{\mathbf{p}})\hat{\mathbf{p}} \cdot \underline{\mathbf{S}}. \quad (12)$$

Since the basis vectors are eigenvectors of $\hat{\mathbf{p}}\cdot\underline{\mathbf{S}}$ with eigenvalue λ, that is, $\hat{\mathbf{p}}\cdot\underline{\mathbf{S}}\underline{e}_{\lambda}=\lambda\underline{e}_{\lambda}^{(\chi)}$, they are also eigenvectors of the position operator with eigenvalue 0, giving $\underline{\mathbf{r}}^{(\chi)}\underline{e}_{\lambda}^{(\chi)}=0$ and thus $\underline{\mathbf{L}}^{(\chi)}\underline{e}_{\lambda}^{(\chi)}=0$ In a basis expansion, $\underline{\mathbf{L}}^{(\chi)}$ just differentiates the coefficient of $\underline{e}_{\lambda}^{(\chi)}$, giving no contribution due to the basis; the operator $\underline{\mathbf{S}}^{(\chi)}$ alone extracts the total angular momentum of the basis.

Restrictions on the uncertainty of the angular momentum of a localized state are imposed by the commutation relations between the components of $\underline{\mathbf{r}}^{(\chi)}$ and $\underline{\mathbf{J}}$, which were found in Ref. [11] to be

$$[\underline{J}_j, \underline{r}_k] = i\epsilon_{jkl}\underline{r}_l - i\lambda(\partial\underline{S}_j^{(\chi)}/\partial p_k). \quad (13)$$

Note that the position operator does not transform as a simple vector because, through its coupling to the spin, a rotation induces a gauge change. For a photon at the origin, $\langle\underline{r}_k\rangle=0$ and the usual relationship between uncertainty and the commutator gives

$$\Delta\underline{J}_j\Delta\underline{r}_k \geqslant \frac{1}{2}\langle|\partial\underline{S}_j^{(\chi)}/\partial p_k|\rangle \quad (14)$$

and

$$\Delta\underline{J}^2\Delta\underline{r}_k \geqslant \sum_j \langle|\underline{J}_j\partial\underline{S}_j^{(\chi)}/\partial p_k|\rangle. \quad (15)$$

When χ is given by Eq. (8) the z component of $\underline{\mathbf{S}}^{(\chi)}$ reduces to $\underline{S}_z^{(-m\phi)}=m\underline{\mathbf{S}}\cdot\hat{\mathbf{p}}$ and within a state space of helicity λ, $\partial\underline{S}_z^{(-m\phi)}/\partial p_k=0$. Thus the photon can simultaneously have a definite position and z component of the total angular momentum. However, it does not have definite x or y component of $\underline{\mathbf{J}}$, and there is no definite value for the magnitude of total angular momentum. Nothing can be known definitely about the values of $\underline{\mathbf{S}}$ or $\underline{\mathbf{L}}$ separately. This is consistent with the expansion (10).

III. TRANSVERSE FIELDS

The transverse basis vectors (6) can be used to express either the ideally localized states given by Eq. (1) or more readily realizable states. Adlard, Pike, and Sarkar [8], for example, constructed single-photon states with arbitrarily fast asymptotic power-law falloff of energy density and photodetection rate and Białynicki-Birula [9] obtained converging or diverging localized states with an arbitrarily fast exponential falloff. An advantage of these latter states is that the falloff rates for the vector potential, the fields, and the Landau-Peierls photon wave function [16] are asymptotically all determined by the same exponential factor, and this avoids the problem that the fields themselves associated with exactly localized states are not localized [17]. The gauge choice $\chi(\theta,\phi)$ affects the angular momentum of the basis states, whether applied to these asymptotically localized states or to the exactly localized states of Ref. [11].

In coordinate space the electric field describing the localized states with helicity λ discussed here can be written as

$$E_\mu(\mathbf{r},t,\lambda) = \int \frac{d^3p}{(2\pi\hbar)^3} f(p)g(\theta)e_{\lambda,\mu}^{(\chi)}(\theta,\phi) \times \exp[i(\mathbf{p}\cdot\mathbf{r} - \text{pct})/\hbar] \quad (16)$$

where $g(\theta)=\sin\theta$ for the localized states considered in [9], while $g(\theta)=1$ in [8,11] giving, with the gauge choice (8),

$$e_{\lambda,\mu}^{(\chi)}(\theta,\phi) = e_{\lambda,\mu}^{(\chi)}(\theta,0)e^{i(m-\mu)\lambda\phi}. \quad (17)$$

To transform to coordinate space we can use spherical polar coordinates and expand in spherical harmonics using

$$\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar) = 4\pi\sum_{l=0}^{\infty}\sum_{n=-l}^{l} i^l Y_l^n(\vartheta,\varphi)Y_l^{n*}(\theta,\phi)j_l(pr/\hbar).$$

Integration over ϕ then gives

$$E_\mu(\mathbf{r},t,\lambda) = h_\mu(\vartheta,r,t,\lambda)\exp[i(m-\mu)\lambda\varphi] \quad (18)$$

with

$$h_\mu(\vartheta,r,t,\lambda) \equiv \frac{1}{\pi\hbar^3}\sum_{l=|m-\mu|}^{\infty} i^l Y_l^{m-\mu}(\vartheta,0) \times \int d(\cos\theta)Y_l^{m-\mu*}(\theta,0)g(\theta)e_{\lambda,\mu}^{(\chi)}(\theta,0) \times \int dp\, p^2 f(p) j_l\left(\frac{pr}{\hbar}\right)\exp(-ipct/\hbar), \quad (19)$$

where the subscript μ implies the corresponding component in the expansion (10). The position-space field components vary as $\exp[i(m-\mu)\lambda\varphi]$, indicating a z component of orbital angular momentum equal to $\hbar l_z=\hbar\lambda(m-\mu)$. Thus the position-space z components of spin, orbital, and total angular momentum are exactly the same as those in momentum space, and all of the specific results discussed above regarding the z component angular momenta apply in position space.

The ϑ dependence can be obtained by expanding the integrand as

$$\sqrt{2\pi}g(\theta)e^{(\chi)}_{\lambda,\mu}(\theta,0) = \sum_{l=|m-\mu|}^{\infty} c_{\mu,l}Y_l^{m-\mu}(\theta,0) \qquad (20)$$

and using the orthogonality of the spherical harmonics with the same $|s_z|$ value. We consider a few examples. If $m=1$ and $g_\lambda=1$ then $c_{0,l}=4/\sqrt{6}Y_1^1\delta_{l,1}$ so that the z component of the field $\sim\sin\vartheta$. For the counterclockwise-rotating component of the field, $c_{1,l}=4/\sqrt{3}Y_0^0\delta_{l,0}+2/\sqrt{6}Y_1^0\delta_{l,1}$, which gives a ϑ-independent term and a $\cos\vartheta$ term. The basis in Ref. [9] implies $m=0$ and $g=\sin\theta$ and gives $c_{0,l}=4/\sqrt{3}Y_0^0\delta_{l,0}-10/\sqrt{8}Y_2^0\delta_{l,2}$. In all cases the field component vanishes on any axis along which the component of $\mathbf{L}$ has a nonzero value. This is true in general since $j_l(0)=0$ for $l>0$ [see Eq. (19)].

The singularity of $\mathbf{a}^{(\chi)}$ discussed in Sec. II is the axis of a vortex. Expression (10) makes explicit the angular momenta of the basis vectors along the direction of the string, and associated clockwise and counterclockwise rotation about it. The polar angle $\theta=0$ identifies the positive z axis and the paraxial limit when describing a beam, while $\theta=\pi$ identifies the negative z axis. If $m=0$, the whole z axis is singular, while if $m=1$, there is no singularity associated with the positive z axis ($l_z=0$), but the negative z axis has $l_z=2\lambda$, that is, it has twice the strength or topological charge. The singularity has just been moved from the positive to the negative z axis. The center of the vortex has zero intensity as discussed above. The orbital angular momentum arises from a bright annular ring about the axis, as witnessed in the $j_l(pr/\hbar)\sin\vartheta$ dependence of the field, and the radius of this ring goes to zero with the parameter describing the spatial extent of the localized photon state.

IV. CONCLUSION

In summary, we have examined the physical meaning of the gauge associated with the photon position operator, and found that position operators with a different geometrical gauge have localized eigenvectors with different angular momenta. The z component of the total angular momentum operator commutes with the position operator, and localized photon states of definite helicity can also have a definite j_z. We find that asymptotically localized photon states are not spherically symmetric fuzzy balls, but can have the screw phase dislocation or optical vortex structure that characterizes Laguerre-Gaussian beams. The situation is analogous to a Dirac monopole and its associated string singularity, and the same mathematical results apply. A geometric gauge transformation can change the total angular momentum of the localized basis states, but a deviation from spherical symmetry remains.

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